

HOMOTOPICALLY EQUIVALENT SIMPLE LOOPS ON 2-BRIDGE SPHERES IN 2-BRIDGE LINK COMPLEMENTS (III)

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ABSTRACT. This is the last of the series of the papers with the aim to give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in a 2-bridge link complement to be homotopic in the link complement. The first paper [7] treated the case of the 2-bridge torus links, and the second paper [8] treated the case of 2-bridge links of slope $n/(2n+1)$ and $(n+1)/(3n+2)$, where $n \geq 2$ is an arbitrary integer. This paper first treats the case of 2-bridge links of slope $n/(mn+1)$ and $(n+1)/((m+1)n+m)$, where $m \geq 3$ is an arbitrary integer, and then the remaining cases by induction.

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1. INTRODUCTION

Let K be a 2-bridge link in S^3 and let S be a 4-punctured sphere in $S^3 - K$ obtained from a 2-bridge sphere of K . The present paper is a continuation of [7] and [8] in the series of the papers with the aim to give a necessary and sufficient condition for two essential simple loops on S to be homotopic in $S^3 - K$. Ahead of this series, the authors [6] gave a complete characterization of those essential simple loops in S which are null-homotopic in $S^3 - K$.

By $[m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$, we denote a rational number with the following continued fraction expansion:

$$[m_1, m_2, \dots, m_k] := \frac{1}{m_1 + \frac{1}{m_2 + \dots + \frac{1}{m_k}}}$$

The first paper [7] of the series treated the case of a 2-bridge link of slope $1/p = [p]$, and the second paper [8] treated the case of a 2-bridge link of slope $n/(2n+1) = [2, n]$ or slope $(n+1)/(3n+2) = [2, 1, n]$. On the other hand, the first half of the present paper treats the case of a 2-bridge link of slope $n/(mn+1) = [m, n]$ or slope $(n+1)/((m+1)n+m) = [m, 1, n]$, where $m \geq 3$ is an arbitrary integer. These five families play special roles in our project in the sense that the treatment of these links form a base step of an inductive proof of the main theorem for a 2-bridge link of general slope $[m_1, m_2, \dots, m_k]$ to which the second half of the present paper contributes, where the induction uses $k \geq 1$ as the parameter.

In the present paper, we also give a complete characterization of those simple loops in the 2-bridge sphere of a hyperbolic 2-bridge link to be peripheral or imprimitive in the link complement (see Theorems 2.6 and 2.7).

These results are used in [9] to show the existence of a variation of McShane's identity for 2-bridge links, thus proving a conjecture proposed in [15]. For an overview of this series of works, we refer the reader to the research announcement [10].

It has been proved by Weinbaum [18] and Appel and Schupp [1] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [4] and references in it). Moreover, it was also shown by Sela [17] and Pr  aux [13] that the word and conjugacy problems for any link group are solvable. A characteristic feature of this series of papers including [6] is that we give a complete answer to special (but also natural) word and conjugacy problems for the groups of 2-bridge links, which form a special (but also important) family of prime alternating links. The key tool used in the proofs is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups.

This paper is organized as follows. In Section 2, we recall basic facts concerning 2-bridge links, and describe the main results of this paper (Main Theorem 2.5 and Theorems 2.6 and 2.7). In Section 3, we recall the upper presentation of a 2-bridge link group, and recall key facts established in [6], [7], and [8] concerning the upper presentation. In Section 4, we establish technical lemmas used for the proofs in Sections 5 and 6. Two special cases of Main Theorem 2.5 namely, the cases of a 2-bridge link of slope $n/(mn+1) = [m, n]$ and a 2-bridge link of slope $(n+1)/((m+1)n+m) = [m, 1, n]$, where $m, n \geq 3$, are treated in Sections 5 and 6, respectively. For the case of a 2-bridge link of general slope, we start with preliminary results in Section 7 and perform transformation so that we may apply the induction as discussed in Section 8. In Section 9, we prove key results for the induction, and finally the proof of Main Theorem 2.5 for the general cases is contained in Section 10. In Section 11, we prove Theorems 2.6 and 2.7.

2. MAIN RESULT

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope r , which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope ∞ and r . The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(\mathbf{S}^2, \mathbf{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where H is the group of isometries of the Euclidean plane \mathbb{R}^2

generated by the π -rotations around the points in the lattice \mathbb{Z}^2 (see [12, Section 3] or [6, Section 2] for details). Let \mathbf{S} be the 4-punctured sphere $\mathbf{S}^2 - \mathbf{P}$ in the link complement $S^3 - K(r)$. Any essential simple loop in \mathbf{S} , up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto \mathbf{S} . The (unoriented) essential simple loop in \mathbf{S} so obtained is denoted by α_s . We also denote by α_s the conjugacy class of an element of $\pi_1(\mathbf{S})$ represented by (a suitably oriented) α_s . Then the *link group* $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(\mathbf{S}) / \langle \langle \alpha_\infty, \alpha_r \rangle \rangle$.

Let \mathcal{D} be the *Farey tessellation*, whose ideal vertex set is identified with $\hat{\mathbb{Q}}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of \mathcal{D} generated by reflections in the edges of \mathcal{D} with an endpoint r , and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞ . Assume that $r \neq \infty$. Then the region, R , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on \mathbb{H}^2 (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R . Suppose that $0 < r < 1$ is a rational number with $r = [m_1, m_2, \dots, m_k]$, where $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Then $I_1(r) = [0, r_1]$ and $I_2(r) = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

The following theorem was established by [12] and [6], which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

Theorem 2.1. (1) [12, Proposition 4.6] *If two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$ -orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) [6, Lemma 7.1] *For any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1(r) \cup I_2(r) \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$ -orbit of s_0 . In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$, then α_s is null-homotopic in $S^3 - K(r)$.*

(3) [6, Main Theorem 2.3] *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$ -orbit of ∞ or r . In particular, if $s \in I_1(r) \cup I_2(r)$, then α_s is not null-homotopic in $S^3 - K(r)$.*

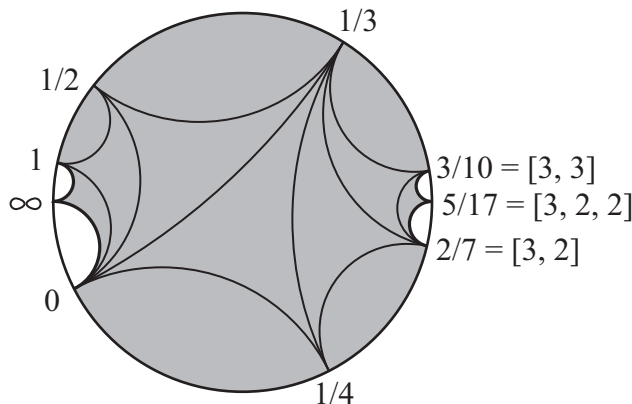


FIGURE 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

Thus the following question naturally arises.

Question 2.2. *Consider a 2-bridge link $K(r)$ with $r \neq \infty$. For two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$, when are the unoriented loops α_s and $\alpha_{s'}$ homotopic in $S^3 - K(r)$?*

If $r = \infty$, then $G(K(\infty))$ is a rank 2 free group, and the work of Komori and Series [5, Theorem 1.2] implies that α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(\infty)$ if and only if s and s' belong to the same orbit of $\hat{\Gamma}_\infty = \Gamma_\infty$.

The purpose of this series of papers is to solve the above question. By Schubert's classification of 2-bridge links [16], we may assume that r is a rational number with $0 \leq r \leq 1/2$. If $r = 0$, then $\hat{\Gamma}_0$ is equal to the group generated by the reflections in the edges of any of \mathcal{D} , and $I_1(0) \cup I_2(0) = \{1\}$, a singleton set. Thus we have nothing to do with Question 2.2, and so we may assume $0 < r \leq 1/2$. In the first paper [7] and the second paper [8] of this series, we gave an answer to the question, respectively, for $r = 1/p$ with $p \geq 2$ and for $r = n/(2n+1) = [2, n]$ or $r = (n+1)/(3n+2) = [2, 1, n]$ with $n \geq 2$. The following theorem summarizes these results.

Theorem 2.3. *The following hold for two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$.*

- (1) [7, Main Theorem 2.7] *Suppose $r = 1/p$, where $p \geq 2$ is an integer. Then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.*

- (2) [8, Main Theorem 2.4] *Suppose $r = n/(2n + 1) = [2, n]$, where $n \geq 2$ is an integer. Then the unoriented loops α_s and $\alpha_{s'}$ are never homotopic in $S^3 - K(r)$.*
- (3) [8, Main Theorem 2.5] *Suppose $r = (n + 1)/(3n + 2) = [2, 1, n]$, where $n \geq 2$ is an integer. Then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if both $r = 3/8$ (i.e., $n = 2$) and the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.*

Since there exist a homeomorphism from $(S^3, K(n/(2n+1)))$ to $(S^3, K(2/(2n+1)))$ and a homeomorphism from $(S^3, K((n+1)/(3n+2)))$ to $(S^3, K(3/(3n+2)))$ both of which send the upper/lower tangles to lower/upper tangles, the following corollary is immediate from Theorem 2.3(2) and (3).

Corollary 2.4. *Suppose $r = 2/(2n + 1) = [n, 2]$ or $r = 3/(3n + 2) = [n, 1, 2]$, where $n \geq 3$ is an integer. Then, for any two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$, the unoriented loops α_s and $\alpha_{s'}$ are never homotopic in $S^3 - K(r)$.*

In the present paper, we solve Question 2.2 for the remaining cases.

Main Theorem 2.5. *Suppose that r is a rational number with $0 < r \leq 1/2$ such that $r \neq 1/n$, $r \neq n/(2n + 1)$, $r \neq 2/(2n + 1)$, $r \neq (n + 1)/(3n + 2)$ and $r \neq 3/(3n + 2)$, where $n \geq 2$ is an integer. Then, for any two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$, the unoriented loops α_s and $\alpha_{s'}$ are never homotopic in $S^3 - K(r)$.*

The proof of the main theorem together with [8, Main Theorems 2.4 and 2.5] implies the following theorems, which give a complete characterization of those simple loops in the 2-bridge sphere of a hyperbolic 2-bridge link to be peripheral or imprimitive in the link complement.

Theorem 2.6. *The following hold for a rational number $s \in I_1(r) \cup I_2(r)$.*

- (1) *Suppose $r = n/(2n + 1) = [2, n]$, where $n \geq 2$ is an integer. Then the loop α_s is peripheral if and only if one of the following holds.*
 - (a) $n = 2$, i.e., $r = 2/5$, and $s = 1/5$ or $s = 3/5$.
 - (b) $s = (n + 1)/(2n + 1)$.
- (2) *Suppose $r = 2/(2n + 1) = [n, 2]$, where $n \geq 3$ is an integer. Then the loop α_s is peripheral if and only if $s = n/(2n + 1)$.*
- (3) *Suppose that r is a rational number with $0 < r \leq 1/2$ such that $r \neq 1/n$, $r \neq n/(2n + 1)$ and $r \neq 2/(2n + 1)$, where $n \geq 2$ is an integer. Then the loop α_s is never peripheral.*

Theorem 2.7. *Suppose that r is a rational number with $0 < r \leq 1/2$ such that $r \neq 1/n$, where $n \geq 2$ is an integer. Then, for a rational number $s \in I_1(r) \cup I_2(r)$, the free homotopy class α_s is primitive with the following exceptions.*

- (1) $r = 2/5$, and $s = 2/7$ or $s = 3/4$. In this case, α_s is the third power of some primitive element in $G(K(r))$.
- (2) $r = 3/7$ and $s = 2/7$. In this case, α_s is the second power of some primitive element in $G(K(r))$.
- (3) $r = 2/7$ and $s = 3/7$. In this case, α_s is the second power of some primitive element in $G(K(r))$.

Here, a closed loop α_s in $S^3 - K(r)$ is said to be *peripheral* if it is homotopic to a loop on a peripheral torus. A loop α_s is said to be *primitive* if there is no element in the 2-bridge link group $G(K(r))$ whose proper power is conjugate to α_s .

We prove the above main theorems by interpreting the situation in terms of combinatorial group theory. In other words, we prove that two words representing the free homotopy classes of α_s and $\alpha_{s'}$ are never conjugate in the 2-bridge link group $G(K(r))$ for any two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$. The key tool used in the proofs is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups.

3. REVIEW OF BASIC RESULTS FROM [6], [7] AND [8]

Throughout this paper, the set $\{a, b\}$ denotes the standard meridian-generator of the rank 2 free group $\pi_1(B^3 - t(\infty))$, which is specified as in [6, Section 3] or [7, Section 3]. For a positive rational number q/p , let $u_{q/p}$ be the word in $\{a, b\}$ representing the (suitably oriented) loop $\alpha_{q/p}$ defined by the following rule (see [6, Lemma 4.7]):

$$u_{q/p} = a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{2p-1}} b^{\varepsilon_{2p}}.$$

Here $\varepsilon_i = (-1)^{\lceil (i-1)q/p \rceil^* - 1}$, with $\lceil t \rceil^*$ the smallest integer greater than t . In fact, $u_{q/p}$ is obtained from a line of slope q/p in $\mathbb{R}^2 - \mathbb{Z}^2$ by reading its intersection with vertical lattice lines (cf. [6, Remark 1]).

The link group $G(K(r))$ with $r > 0$ has the following presentation, called the *upper presentation*:

$$\begin{aligned} G(K(r)) &= \pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r \rangle \rangle \\ &\cong F(a, b) / \langle \langle u_r \rangle \rangle \cong \langle a, b \mid u_r \rangle. \end{aligned}$$

3.1. S - and T -sequences of slope r . We recall the definition of the sequences $S(r)$ and $T(r)$ and the cyclic sequences $CS(r)$ and $CT(r)$ of slope r defined in [6], all of which are read from the single relator u_r of the upper presentation of $G(K(r))$, and review several important properties of these sequences from [6] so that we can adopt small cancellation theory. To this end we fix some definitions and notation. Let X be a set. By a *word* in X , we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i -th letter* of the word. For two words u, v in X , by $u \equiv v$ we denote the *visual equality* of u and v , meaning that if $u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \cdots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v . Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v) . In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u .

Definition 3.1. (1) Let v be a nonempty reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \cdots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S -sequence of v* .

(2) Let (v) be a nonempty cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := ((|v_1|, |v_2|, \dots, |v_t|))$ is called the *cyclic S -sequence of (v)* . Here, the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A nonempty reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v . A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

The following lemma is obvious from the definition.

Lemma 3.2 ([6, Proposition 4.1(1)]). *An alternating word in $\{a, b\}$ is completely determined by the initial letter and the associated S -sequence.*

Definition 3.3. For a rational number r with $0 < r \leq 1$, let $G(K(r)) = \langle a, b \mid u_r \rangle$ be the upper presentation. Then the symbol $S(r)$ (resp. $CS(r)$) denotes the S -sequence $S(u_r)$ of u_r (resp. cyclic S -sequence $CS(u_r)$ of (u_r)), which is called the S -sequence of slope r (resp. the cyclic S -sequence of slope r).

Lemma 3.4 ([6, Lemma 5.2]). *Let w be an arbitrary cyclic permutation of the single relator u_r of the group presentation of $G(K(r))$. Then the set*

$$\{\text{the initial letter of } w' \mid (w') \equiv (u_r^{\pm 1}) \text{ and } S(w') = S(w)\}$$

equals $\{a, a^{-1}, b, b^{-1}\}$.

In the remainder of this paper unless specified otherwise, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction:

$$r = [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. For brevity, we write m for m_1 .

Lemma 3.5 ([6, Proposition 4.3]). *The following hold.*

- (1) *Suppose $k = 1$, i.e., $r = 1/m$. Then $S(r) = (m, m)$.*
- (2) *Suppose $k \geq 2$. Then each term of $S(r)$ is either m or $m + 1$, and $S(r)$ begins with $m + 1$ and ends with m . Moreover, the following hold.*
 - (a) *If $m_2 = 1$, then no two consecutive terms of $S(r)$ can be (m, m) , so there is a sequence of positive integers (t_1, t_2, \dots, t_s) such that*

$$S(r) = (t_1 \langle m + 1 \rangle, m, t_2 \langle m + 1 \rangle, m, \dots, t_s \langle m + 1 \rangle, m).$$

Here, the symbol " $t_i \langle m + 1 \rangle$ " represents t_i successive $m + 1$'s.

- (b) *If $m_2 \geq 2$, then no two consecutive terms of $S(r)$ can be $(m + 1, m + 1)$, so there is a sequence of positive integers (t_1, t_2, \dots, t_s) such that*

$$S(r) = (m + 1, t_1 \langle m \rangle, m + 1, t_2 \langle m \rangle, \dots, m + 1, t_s \langle m \rangle).$$

Here, the symbol " $t_i \langle m \rangle$ " represents t_i successive m 's.

Definition 3.6. If $k \geq 2$, the symbol $T(r)$ denotes the sequence (t_1, t_2, \dots, t_s) in Lemma 3.5, which is called the T -sequence of slope r . The symbol $CT(r)$ denotes the cyclic sequence represented by $T(r)$, which is called the cyclic T -sequence of slope r .

Lemma 3.7 ([6, Proposition 4.4]). *Let \tilde{r} be the rational number defined as*

$$\tilde{r} = \begin{cases} [m_3, \dots, m_k] & \text{if } m_2 = 1; \\ [m_2 - 1, m_3, \dots, m_k] & \text{if } m_2 \geq 2. \end{cases}$$

Then we have

$$T(r) = \begin{cases} S(\tilde{r}) & \text{if } m_2 = 1; \\ \overleftarrow{S}(\tilde{r}) & \text{if } m_2 \geq 2, \end{cases}$$

where $\overleftarrow{S}(\tilde{r})$ denotes the sequence obtained from $S(\tilde{r})$ reversing its order.

Proposition 3.8 ([6, Proposition 4.5]). *The sequence $S(r)$ has a decomposition (S_1, S_2, S_1, S_2) which satisfies the following.*

- (1) *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i . (Here, S_1 is empty if $k = 1$.)*
- (2) *Each S_i occurs only twice in the cyclic sequence $CS(r)$.*
- (3) *S_1 begins and ends with $m + 1$.*
- (4) *S_2 begins and ends with m .*

Corollary 3.9 ([6, Corollary 4.6]). *The cyclic S -sequence $CS(r)$ is symmetric, i.e., the cyclic sequence obtained from $CS(r)$ by reversing its cyclic order is equivalent to $CS(r)$ (as a cyclic sequence). In particular, in Lemma 3.7, we actually have*

$$CT(r) = CS(\tilde{r}).$$

Remark 3.10. By using the fact that u_r is obtained from of the line of slope r in $\mathbb{R}^2 - \mathbb{Z}^2$ by reading its intersection with the vertical lattice lines, we see that the slope $s = q/p$ is recovered from $CS(s) = ((S_1, S_2, S_1, S_2))$ by the rule that p is the sum of the components of S_1 and S_2 whereas q is the sum of the lengths of S_1 and S_2 .

Lemma 3.11 ([6, Proof of Proposition 4.5]). *Let \tilde{r} be the rational number defined as in Lemma 3.7. Also let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ and $S(r) = (S_1, S_2, S_1, S_2)$ be decompositions described as in Proposition 3.8. Then the following hold.*

- (1) *If $m_2 = 1$ and $k = 3$, then $T_1 = \emptyset$, $T_2 = (m_3)$, and $S_1 = (m_3 \langle m + 1 \rangle)$, $S_2 = (m)$.*
- (2) *If $m_2 = 1$ and $k \geq 4$, then $T_1 = (t_1, \dots, t_{s_1})$, $T_2 = (t_{s_1+1}, \dots, t_{s_2})$, and*

$$S_1 = (t_1 \langle m + 1 \rangle, m, t_2 \langle m + 1 \rangle, \dots, t_{s_1-1} \langle m + 1 \rangle, m, t_{s_1} \langle m + 1 \rangle),$$

$$S_2 = (m, t_{s_1+1} \langle m + 1 \rangle, m, \dots, m, t_{s_2} \langle m + 1 \rangle, m).$$

- (3) If $m_2 \geq 2$ and $k = 2$, then $T_1 = \emptyset$, $T_2 = (m_2 - 1)$, and $S_1 = (m + 1)$,
 $S_2 = ((m_2 - 1)\langle m \rangle)$.
- (4) If $m_2 \geq 2$ and $k \geq 3$, then $T_1 = (t_1, \dots, t_{s_1})$, $T_2 = (t_{s_1+1}, \dots, t_{s_2})$, and
 $S_1 = (m + 1, t_{s_1+1}\langle m \rangle, m + 1, \dots, m + 1, t_{s_2}\langle m \rangle, m + 1)$,
 $S_2 = (t_1\langle m \rangle, m + 1, t_2\langle m \rangle, \dots, t_{s_1-1}\langle m \rangle, m + 1, t_{s_1}\langle m \rangle)$.

The following is a refinement of [6, Lemma 7.3 and Remark 5].

Proposition 3.12 ([8, Proposition 3.12]). *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 3.8. For a rational number s with $0 < s \leq 1$, suppose that the cyclic S -sequence $CS(s)$ contains both S_1 and S_2 as subsequences. Then $s \notin I_1(r) \cup I_2(r)$.*

In the above proposition (and throughout this paper), we mean by a *subsequence* a subsequence without leap. Namely a sequence (a_1, a_2, \dots, a_p) is called a *subsequence* of a cyclic sequence, if there is a sequence (b_1, b_2, \dots, b_n) representing the cyclic sequence such that $p \leq n$ and $a_i = b_i$ for $1 \leq i \leq p$.

3.2. Small cancellation theory. We now recall the small cancellation conditions for the 2-bridge link groups established in [6]. Let $F(X)$ be the free group with basis X . A subset R of $F(X)$ is said to be *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R .

Definition 3.13. Suppose that R is a symmetrized subset of $F(X)$. A nonempty word v is called an *R -piece*, or a piece in brief, if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv vc_1$ and $w_2 \equiv vc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [11]).

- (1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \geq p$.
- (2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair $(i \bmod n)$, if $n < q$, then at least one of the products $w_1w_2, \dots, w_{n-1}w_n, w_nw_1$ is freely reduced without cancellation.

The following proposition enables us to apply the small cancellation theory to our problem.

Proposition 3.14 ([6, Theorem 5.1]). *Suppose that r is a rational number with $0 < r < 1$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$. Then R satisfies $C(4)$ and $T(4)$.*

We recall a key fact concerning the cyclic word (u_r) , which is used in the proofs of the main theorems

Definition 3.15. For a positive integer n , a nonempty subword w of the cyclic word (u_r) is called a *maximal n -piece* if w is a product of n pieces and if any subword w' of u_r which properly contains w as an *initial* subword is not a product of n -pieces.

Lemma 3.16 ([6, Corollary 5.4(2)]). *Suppose that r is a rational number such that $0 < r < 1$ and $r \neq 1/p$ for any integer $p \geq 2$. Let u_r be the single relator of the upper presentation of $G(K(r))$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 3.8. Decompose*

$$u_r \equiv v_1 v_2 v_3 v_4,$$

where $S(v_1) = S(v_3) = S_1$ and $S(v_2) = S(v_4) = S_2$. Let v_{ib}^* be the maximal proper initial subword of v_i , i.e., the initial subword of v_i such that $|v_{ib}^*| = |v_i| - 1$ ($i = 1, 2, 3, 4$). Then the following hold, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.

- (1) The following is the list of all maximal 1-pieces of (u_r) , arranged in the order of the position of the initial letter:

$$v_{1b}^*, v_{1e}v_2, v_2v_{3b}^*, v_{2e}v_{3b}^*, v_{3b}^*, v_{3e}v_4, v_4v_{1b}^*, v_{4e}v_{1b}^*.$$

- (2) The following is the list of all maximal 2-pieces of (u_r) , arranged in the order of the position of the initial letter:

$$v_1v_2, v_{1e}v_2v_{3b}^*, v_2v_3v_4, v_{2e}v_3v_4, v_3v_4, v_{3e}v_4v_{1b}^*, v_4v_1v_2, v_{4e}v_1v_2.$$

Corollary 3.17 ([8, Corollary 3.17]). (1) A subword w of the cyclic word $(u_r^{\pm 1})$ is a piece if and only if $S(w)$ does not contain S_1 as a subsequence and does not contain S_2 in its interior, i.e., $S(w)$ does not contain a subsequence (ℓ_1, S_2, ℓ_2) for some $\ell_1, \ell_2 \in \mathbb{Z}_+$.

(2) For a subword w of the cyclic word $(u_r^{\pm 1})$, if $S(w)$ either contains (S_1, S_2) as a initial subsequence or contains (S_2, S_1) as a proper terminal subsequence, then w is not a product of two pieces.

We recall the following well-known classical result in combinatorial group theory.

Lemma 3.18 ([11, Lemmas V.5.1 and V.5.2]). *Suppose $G = \langle X \mid R \rangle$ with R being symmetrized. Let u, v be two cyclically reduced words in X which are not trivial in G and which are not conjugate in $F(X)$. Then u and v represent conjugate elements in G if and only if there exists a reduced annular R -diagram*

M such that u is an outer boundary label and v^{-1} is an inner boundary label of M .

The following theorem giving a geometric description of the annular diagrams forms a foundation of the whole papers in this series.

Theorem 3.19 ([7, Theorem 4.11 and Corollary 4.13]). *Suppose that r is a rational number with $0 < r < 1$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 3.8. Suppose that M is a reduced annular R -diagram such that*

- (i) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically reduced;*
- (ii) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically alternating;*
- (iii) *the cyclic S -sequences of the cyclic words $(\phi(\alpha))$ and $(\phi(\delta))$ do not contain (S_1, S_2) nor (S_2, S_1) as a subsequence,*

where α and δ are, respectively, arbitrary outer and inner boundary cycles of M . Let the outer and inner boundaries of M be denoted by σ and τ , respectively. Then the following hold.

- (1) *The outer and inner boundaries σ and τ are simple, i.e., they are homeomorphic to the circle, and there is no edge contained in $\sigma \cap \tau$.*
- (2) *$d_M(v) = 2$ or 4 for every vertex $v \in \partial M$. Moreover, on both σ and τ , vertices of degree 2 appear alternately with vertices of degree 4.*
- (3) *$d_M(v) = 4$ for every vertex $v \in M - \partial M$.*
- (4) *$d_M(D) = 4$ for every face $D \in M$.*

In particular, Figure 2(a) illustrates the only possible type of the outer boundary layer of M , while Figure 2(b) illustrates the only possible type of whole M . (The number of faces per layer and the number of layers are variable.)

3.3. Hypotheses A, B and C. We set up Hypotheses A, B and C, and recall several technical lemmas.

Hypothesis A. Let r be a rational number such that $0 < r < 1$ and $r \neq 1/p$ for any integer $p \geq 2$. For two distinct elements $s, s' \in I_1(r) \cup I_2(r)$, suppose that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$. Then u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(K(r))$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be the decomposition as in Proposition 3.8. Due to Lemma 3.18, there is a reduced annular R -diagram M such that u_s and $u_{s'}^{\pm 1}$ are, respectively, outer and inner boundary labels of M . Then we see

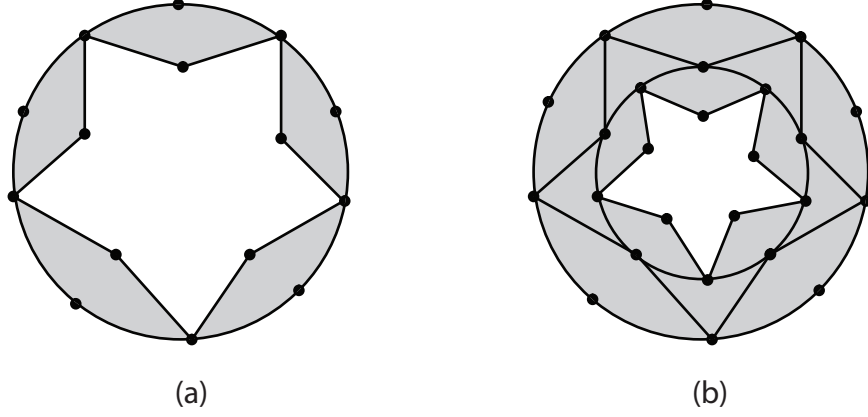


FIGURE 2.

from Proposition 3.12 that M satisfies the three hypotheses (i), (ii) and (iii) of Theorem 3.19.

Let J be the outer boundary layer of M (see Figure 2(a)). Also let α and δ be, respectively, the outer and inner boundary cycles of J starting from v_0 , where v_0 is a vertex lying in both the outer and inner boundaries of J . Here, recall from [7, Convention 4.6] that α is read clockwise and δ is read counterclockwise. Let $\alpha = e_1, e_2, \dots, e_{2t}$ and $\delta^{-1} = e'_1, e'_2, \dots, e'_{2t}$ be the decompositions into oriented edges in ∂J . Then clearly for each $i = 1, \dots, t$, there is a face D_i of J such that $e_{2i-1}, e_{2i}, e'_{2i-1}, e'_{2i}$ are consecutive edges in a boundary cycle of D_i . We denote the path e_{2i-1}, e_{2i} by ∂D_i^+ and the path e'_{2i-1}, e'_{2i} by ∂D_i^- . In particular, if $J \subsetneq M$ (see Figure 2(b)), then, for each $i = 1, \dots, t$, there is a face D'_i in $M - J$ such that e'_{2i} and e'_{2i+1} are two consecutive edges in $\partial D'_i \cap \delta^{-1}$. Here the indices for the 2-cells are considered modulo t , and the indices for the edges are considered modulo $2t$.

Lemma 3.20 ([8, Lemma 4.1]). *Under Hypothesis A, both of the following hold for every i .*

- (1) *None of $S(\phi(e_{2i-1}))$, $S(\phi(e_{2i}))$, $S(\phi(e'_{2i}))$ and $S(\phi(e'_{2i-1}))$ contains S_1 as a subsequence.*
- (2) *None of $S(\phi(e_{2i-1}))$, $S(\phi(e_{2i}))$, $S(\phi(e'_{2i}))$ and $S(\phi(e'_{2i-1}))$ contains a subsequence of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$.*

Lemma 3.21 ([8, Lemma 4.2]). *Under Hypothesis A, only one of the following holds for each face D_i of J .*

- (1) *Both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ contain S_1 as their subsequence.*

- (2) Both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ contain subsequences of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$.

Lemma 3.22 ([8, Lemma 4.3]). *Under Hypothesis A, only one of the following holds.*

- (1) For every face D_i of J , $S(\phi(\partial D_i^+))$ contains S_1 as its subsequence.
(2) For every face D_i of J , $S(\phi(\partial D_i^+))$ contains a subsequence of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$.

Hypothesis B. Suppose under Hypothesis A that Lemma 3.22(1) holds, namely, for every face D_i of J , suppose that $S(\phi(\partial D_i^+))$ contains S_1 as its subsequence. Then we can decompose the word $\phi(\alpha)$ (clearly $(u_s) \equiv (\phi(\alpha))$) into

$$\phi(\alpha) \equiv y_1 w_1 z_1 y_2 w_2 z_2 \cdots y_t w_t z_t,$$

where $\phi(\partial D_i^+) \equiv \phi(e_{2i-1} e_{2i}) \equiv y_i w_i z_i$, y_i and z_i may be empty, $S(w_i) = S_1$, and where $S(y_i w_i z_i) = (S(y_i), S_1, S(z_i))$ (here $S(y_i)$ and $S(z_i)$ are possibly empty), for every i . By Lemma 3.21, we also have the decomposition of the word $\phi(\delta^{-1})$ as follows (clearly $(u_s^{\pm 1}) \equiv (\phi(\delta^{-1}))$ if $J = M$):

$$\phi(\delta^{-1}) \equiv y'_1 w'_1 z'_1 y'_2 w'_2 z'_2 \cdots y'_t w'_t z'_t,$$

where $\phi(\partial D_i^-) \equiv \phi(e'_{2i-1} e'_{2i}) \equiv y'_i w'_i z'_i$, y'_i and z'_i may be empty, $S(w'_i) = S_1$, and where $S(y'_i w'_i z'_i) = (S(y'_i), S_1, S(z'_i))$ (here $S(y'_i)$ and $S(z'_i)$ are possibly empty), for every i . Then $S(y_i'^{-1} y_i) = S(z_i z_i'^{-1}) = S_2$ for every i .

Hypothesis C. Suppose under Hypothesis A that Lemma 3.22(2) holds, namely, for every face D_i of J , suppose that $S(\phi(\partial D_i^+))$ contains a subsequence of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$. Then we can decompose the word $\phi(\alpha)$ (clearly $(u_s) \equiv (\phi(\alpha))$) into

$$\phi(\alpha) \equiv y_1 w_1 z_1 y_2 w_2 z_2 \cdots y_t w_t z_t,$$

where $\phi(\partial D_i^+) \equiv \phi(e_{2i-1} e_{2i}) \equiv y_i w_i z_i$, y_i and z_i are nonempty words, $S(w_i) = S_2$, and where $S(y_i w_i z_i) = (S(y_i), S_2, S(z_i))$, for every i . By Lemma 3.21, we also have the decomposition of the word $\phi(\delta^{-1})$ as follows (clearly $(u_s^{\pm 1}) \equiv (\phi(\delta^{-1}))$ if $J = M$):

$$\phi(\delta^{-1}) \equiv y'_1 w'_1 z'_1 y'_2 w'_2 z'_2 \cdots y'_t w'_t z'_t,$$

where $\phi(\partial D_i^-) \equiv \phi(e'_{2i-1} e'_{2i}) \equiv y'_i w'_i z'_i$, y'_i and z'_i are nonempty words, $S(w'_i) = S_2$, and where $S(y'_i w'_i z'_i) = (S(y'_i), S_2, S(z'_i))$, for every i . Then $S(y_i'^{-1} y_i) = S(z_i z_i'^{-1}) = S_1$ for every i .

The following notation will be used throughout the remainder of this paper.

Notation 3.23. Let v be a reduced word in $\{a, b\}$. If v is not an empty word, then, by v_b and v_e , respectively, we denote a beginning subword and an ending subword of v such that $|v_b|$ is the first term of the sequence $S(v)$ and $|v_e|$ is the last term of $S(v)$. On the other hand, if v is empty, then v_b and v_e are also empty words. If v is represented as a word with suffix, say y_i , then v_b and v_e are denoted by $y_{i,b}$ and $y_{i,e}$, respectively. (Though similar symbols, v_{ib} and v_{ie} ($1 \leq i \leq 4$), are used in different meanings in Lemma 3.16, we believe this does not cause any confusion, because these symbols are not used in the remainder of this paper.)

4. TECHNICAL LEMMAS FOR $r = [m, n]$ OR $r = [m, 1, n]$ WITH $m, n \geq 3$

Throughout this section, we assume that Hypothesis A holds. Then by Lemma 3.22, either Lemma 3.22(1) or Lemma 3.22(2) holds, that is, either Hypothesis B or Hypothesis C holds. Accordingly as Hypothesis B or Hypothesis C holds, we shall establish several technical lemmas used for the proof of Main Theorem 2.5 for $r = [m, n]$ and $r = [m, 1, n]$ in Sections 5 and 6, respectively.

4.1. The case when Hypothesis B holds. We first assume that Hypothesis B holds. We begin with the following remark before introducing technical lemmas concerning the cyclic sequence $CS(\phi(\alpha)) = CS(u_s) = CS(s)$.

Remark 4.1. (1) If $r = [m, n]$, where $m, n \geq 3$ are integers, then, by Lemma 3.11(3), $CS(r) = ((m+1, (n-1)\langle m \rangle, m+1, (n-1)\langle m \rangle))$, where $S_1 = (m+1)$ and $S_2 = ((n-1)\langle m \rangle)$. So, in Hypothesis B, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell_1, n_1\langle m \rangle, m+1, n_2\langle m \rangle, \ell_2)$, where $0 \leq \ell_1, \ell_2 \leq m-1$ and $0 \leq n_1, n_2 \leq n-1$ are integers such that if $n_j = n-1$ then ℓ_j is necessarily 0 for $j = 1, 2$. In particular, $S(y_{i,b}) = (\ell)$ with $1 \leq \ell \leq m$, unless y_i is an empty word. The same is true for $S(z_{i,e})$, $S(y'_{i,b})$ and $S(z'_{i,e})$.

(2) If $r = [m, 1, n]$, where $m, n \geq 3$ are integers, then, by Lemma 3.11(1), $CS(r) = ((n\langle m+1 \rangle, m, n\langle m+1 \rangle, m))$, where $S_1 = (n\langle m+1 \rangle)$ and $S_2 = (m)$. So, in Hypothesis B, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell_1, n\langle m+1 \rangle, \ell_2)$, where $0 \leq \ell_1, \ell_2 \leq m$ are integers. In particular, $S(w_{i,b}) = S(w_{i,e}) = (m+1)$ and $S(w'_{i,b}) = S(w'_{i,e}) = (m+1)$.

Lemma 4.2 (cf. [8, Lemma 4.6]). *Let $r = [m, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, suppose that v is a subword of the cyclic word represented by $\phi(\alpha) \equiv y_1 w_1 z_1 y_2 w_2 z_2 \cdots y_t w_t z_t$ such that v corresponds to a component of $CS(\phi(\alpha)) = CS(s)$. Then, after a cyclic shift of indices, v is equal to one of*

the following subwords:

$$z_{0,e}w_1w_2 \cdots w_qy_{q+1,b}, \quad z_{0,e}w_1w_2 \cdots w_q, \quad w_1w_2 \cdots w_qy_{q+1,b}, \quad w_1w_2 \cdots w_q,$$

where $q \in \mathbb{Z}_+ \cup \{0\}$ in the first three cases and $q \in \mathbb{Z}_+$ in the last case. In each of the above, the “intermediate subwords” are empty; to be precise, when we say that $z_{0,e}w_1w_2 \cdots w_qy_{q+1,b}$, for example, is a subword of (u_s) , we assume that y_1, z_iy_{i+1} ($1 \leq i \leq q-1$) and z_q are empty words.

Throughout the remainder of this paper, we will assume the following convention.

Convention 4.3. In Figures 3–28, except for Figure 15, the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words, and a dot represents a vertex whose position is clearly identified. Also an Arabic number represents the length of the corresponding positive (or negative) word. In Figures 11–12 and 17–19, the label S_1 or S_2 on an oriented segment means that the S -sequence of the corresponding word is equal to S_1 or S_2 accordingly.

Lemma 4.4. Let $r = [m, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, the following hold for every i .

- (1) $S(z_{i,e}y_{i+1,b}) \neq (m+d)$ for any integer d with $1 \leq d \leq m-1$.
- (2) $S(w_iz_iy_{i+1,b}) \neq (m+1+d)$ and $S(z_{i,e}y_{i+1}w_{i+1}) \neq (m+1+d)$ for any integer d with $1 \leq d \leq m-1$.
- (3) $S(w_iz_iy_{i+1,b}) \neq (2m+1)$ and $S(z_{i,e}y_{i+1}w_{i+1}) \neq (2m+1)$.
- (4) $S(w_iz_iy_{i+1}w_{i+1}) \neq (2m+2)$.
- (5) $S(w_iz_iy_{i+1}w_{i+1}) \neq (m+1, m+1)$.

Proof. (1) Suppose on the contrary that the assertion does not hold. Without loss of generality, we may assume that $S(z_{1,e}y_{2,b}) = (m+d)$ for some $1 \leq d \leq m-1$. Then, since $0 \leq |z_{1,e}|, |y_{2,b}| \leq m$, we have $|z_{1,e}| = k$ and $|y_{2,b}| = m+d-k$ for some k with $d \leq k \leq m$. Here, we assume $d \leq k \leq m-1$ and hence $|z'_{1,e}| = m-k \neq 0$. (The case when $k = m$ and hence $|z'_{1,e}| = 0$ can be treated similarly.) Then J is locally as illustrated in Figure 3(a) which follows Convention 4.3. The numbers k and $m+d-k$ near the upper boundary represent the lengths of the words $z_{1,e}$ and $y_{2,b}$, respectively, whereas the numbers $m-k$ and $k-d$ near the lower boundary represent the lengths of the words $z'_{1,e}$ and $y'_{2,b}$, respectively, and the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words.

Suppose first that $J = M$. Then we see from Figure 3(a) that $CS(\phi(\delta^{-1})) = CS(u_s^{\pm 1}) = CS(s')$ involves a term $(m-k) + (k-d) = m-d$. Moreover,

$CS(s')$ also involves a term of the form $m + 1 + c$ with $c \in \mathbb{Z}_+ \cup \{0\}$, because $S(w'_1) = S_1 = (m + 1)$. Since $m - d \leq m - 1$, this is a contradiction to Lemma 3.5 which says that either $CS(s')$ is equal to $((\ell, \ell))$ or $CS(s')$ consists of ℓ and $\ell + 1$ for some $\ell \in \mathbb{Z}_+$. Suppose next that $J \subsetneq M$. Since none of $S(\phi(e'_1))$ and $S(\phi(e'_2))$ contains $S_1 = (m + 1)$ as a subsequence (see Lemma 3.20(1)), we see that the initial vertex of e'_2 lies in the interior of the segment of ∂D_1^- corresponding to $S_1 = (m + 1)$. Similarly, the terminal vertex of e'_3 lies in the interior of the segment of ∂D_2^- corresponding to $S_1 = (m + 1)$. Hence, we see from Figure 3(b) that $S(\phi(e'_2 e'_3))$ contains a subsequence of the form $(\ell_1, m - d, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. This yields that a term $m - d$ occurs in $CS(\phi(\partial D'_1)) = CS(r) = ((m + 1, (n - 1)\langle m \rangle, m + 1, (n - 1)\langle m \rangle))$, which is obviously a contradiction.

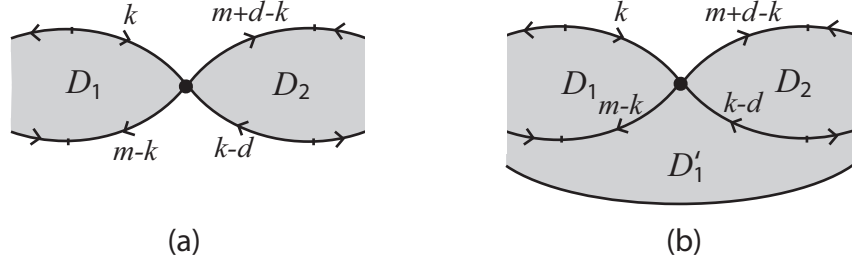


FIGURE 3. Lemma 4.4(1) where $S(z_{1,e}y_{2,b}) = (k + (m + d - k))$.

(2) Suppose on the contrary that $S(w_1 z_1 y_{2,b}) = (m + 1 + d)$ for some $1 \leq d \leq m - 1$. (The other case is treated similarly.) Then, since w_1 and z_1 have different signs when z_1 is nonempty and since $|w_1| = m + 1$ and $0 \leq |y_{2,b}| \leq m$, the only possibility is that $|z_1| = 0$, $|y_{2,b}| = d$ and $S(w_1 y_{2,b}) = (m + 1 + d)$. If $J = M$, then we see from Figure 4(a) that $CS(s')$ involves both a term $m - d$ and a term of the form $m + 1 + c$ with $c \in \mathbb{Z}_+ \cup \{0\}$. Since $m - d \leq m - 1$, this is a contradiction to Lemma 3.5. On the other hand, if $J \subsetneq M$, then we see, by using Lemma 3.20(1) as in the proof of Lemma 4.4(1), that $S(\phi(e'_2 e'_3))$ contains a subsequence of the form $(\ell_1, m - d, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ (see Figure 4(b)). This implies that $CS(\phi(\partial D'_1)) = CS(r)$ has a term $m - d$, a contradiction.

(3) Suppose on the contrary that $S(z_{1,e}y_2 w_2) = (2m + 1)$. (The other case is treated similarly.) Then $|z_{1,e}| = m$ and $|y_2| = 0$. We note that $z_1 = z_{1,e}$. Otherwise, $S(w_1 z_1)$ contains a subword $(m + 1, m, m)$ and hence $CS(\phi(\alpha)) = CS(s)$ contains a term m and $2m + 1$, a contradiction to Lemma 3.5. Hence, we see that $CS(w'_1 z'_1 y'_2 w'_2)$ contains a term $2m$ (see Figure 5(a)). If $J = M$,

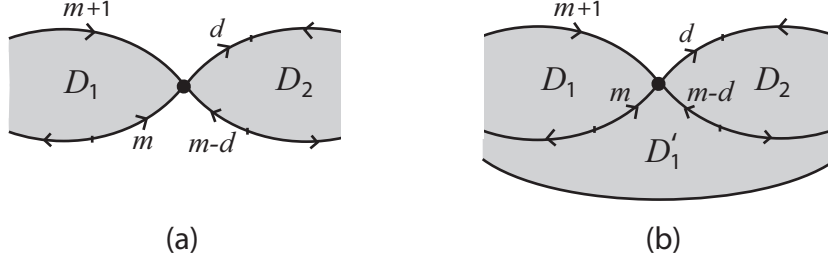


FIGURE 4. Lemma 4.4(2) where $S(w_1 z_1 y_2, b) = ((m+1) + 0 + d)$.

then we see from Figure 5(a) that $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term m and a term $2m$. Since $m \geq 3$ implying that $m+2 \leq 2m$, this gives a contradiction to Lemma 3.5. On the other hand, if $J \subsetneq M$, then we see, by using Lemma 3.20(1) as in the proof of Lemma 4.4(1), that $S(\phi(e'_2 e'_3))$ is of the form $(\ell_1, 2m, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. This implies that a term $2m$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction.

(4) This can be proved by an argument parallel to the proof of (3) (see Figure 5(b)).

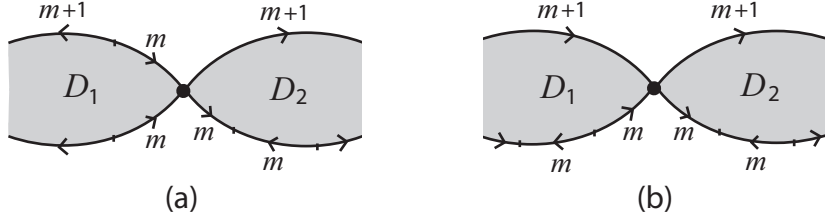


FIGURE 5. (a) Lemma 4.4(3) where $S(z_{1,e} y_2 w_2) = (m+0+(m+1))$, and (b) Lemma 4.4(4) where $S(w_1 z_1 y_2 w_2) = ((m+1)+0+0+(m+1))$.

(5) Suppose on the contrary that $S(w_1 z_1 y_2 w_2) = (m+1, m+1)$. Then $|z_1| = |y_2| = 0$ and $S(w_1 w_2) = (m+1, m+1)$. If $J = M$ (see Figure 6(a)), then $CS(s')$ contains both a subsequence $((n-1)\langle m \rangle)$ and a term of the form $m+1+c$ with $c \in \mathbb{Z}_+ \cup \{0\}$. Here, if $c = 0$, then $s' \notin I_1(r) \cup I_2(r)$ by Proposition 3.12, contradicting the hypothesis of the theorem, while if $c > 0$, then we have a contradiction to Lemma 3.5. On the other hand, if $J \subsetneq M$ (see Figure 6(b)), then we see, by using Lemma 3.20(1) as in the

proof of Lemma 4.4(1), that $S(\phi(e'_2 e'_3))$ is of the form $(\ell_1, 2(n-1)\langle m \rangle, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. This implies that a subsequence $(2(n-1)\langle m \rangle)$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction. \square

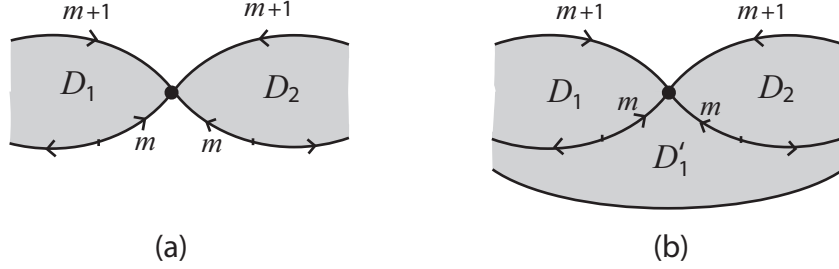


FIGURE 6. Lemma 4.4(5) where $S(w_1 z_1 y_2 w_2) = (m+1, m+1)$.

Lemma 4.5. *Let $r = [m, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, the following hold.*

- (1) *No two consecutive terms of $CS(s)$ can be $(m+1, m+1)$.*
- (2) *No term of $CS(s)$ can be of the form $m+1+d$ except $2m$, where $d \in \mathbb{Z}_+$.*
- (3) *No two consecutive terms of $CS(s)$ can be $(2m, 2m)$.*

Proof. (1) Suppose on the contrary that $CS(\phi(\alpha)) = CS(u_s) = CS(s)$ contains $(m+1, m+1)$ as a subsequence. Let $v = v'v''$ be a subword of the cyclic word (u_s) corresponding to a subsequence $(m+1, m+1)$, where $S(v') = S(v'') = (m+1)$. By using Lemma 4.2 and the facts that $0 \leq |z_{i,e}|, |y_{i,b}| \leq m$ and $|w_i| = m+1$, we see that one of the following holds after a shift of indices.

- (i) $(v', v'') = (z_{1,e} y_{2,b}, w_2)$, where $S(z_{1,e} y_{2,b}) = (m+1)$.
- (ii) $(v', v'') = (w_1, z_{1,e} y_{2,b})$, where $S(z_{1,e} y_{2,b}) = (m+1)$.
- (iii) $(v', v'') = (w_1, w_2)$.

However, (i) and (ii) are impossible by Lemma 4.4(1), and (iii) is impossible by Lemma 4.4(5).

(2) Suppose on the contrary that $CS(s)$ contains a term $m+1+d$ except $2m$. Let v be a subword of the cyclic word (u_s) corresponding to a term $m+1+d$ except $2m$. By using Lemma 4.2 and the facts that $0 \leq |z_{i,e}|, |y_{i,b}| \leq m$ and $|w_i| = m+1$, we see that one of the following holds after a cyclic shift of indices.

- (i) $v = z_{0,e} y_{1,b}$ with $|z_{0,e}|, |y_{1,b}| \neq 0$.
- (ii) v contains $z_{0,e} w_1$ with $|z_{0,e}| \neq 0$.

- (iii) v contains $w_1 y_{2,b}$ with $|y_{2,b}| \neq 0$.
- (iv) v contains $w_1 w_2$ with $|z_1| = |y_2| = 0$.

However, (i) is impossible by Lemma 4.4(1). By Lemma 4.4(2), $|z_{0,e}| = m$ provided (ii) occurs, and $|y_{2,b}| = m$ provided (iii) occurs. Hence (ii) and (iii) are impossible by Lemma 4.4(3). Also (iv) is impossible by Lemma 4.4(4).

(3) Suppose on the contrary that $CS(s)$ contains $(2m, 2m)$ as a subsequence. Let $v = v'v''$ be a subword of the cyclic word (u_s) corresponding to a subsequence $(2m, 2m)$, where $S(v') = S(v'') = (2m)$. If v' contains w_i for some i , then we see, by using Lemma 4.2 and the identity $|w_i| = m + 1$, that either $v' = w_i z_i y_{i+1,b}$ with $(|z_i|, |y_{i+1,b}|) = (0, m - 1)$ or $v' = z_{i-1,e} y_i w_i$ with $(|z_{i-1}|, |y_i|) = (m - 1, 0)$. However both cases are impossible by Lemma 4.4(2). Thus v' cannot contain w_i . Since $S(v') = (2m)$ is a component of $CS(s)$, this implies that v' is disjoint from w_i for every i . The same conclusion also holds for v'' , and hence for $v = v'v''$. Thus v is a subword of $z_i y_{i+1}$ for some i . But then $S(v)$ contains a term $2m$ at most once, a contradiction. \square

Corollary 4.6. *Let $r = [m, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, $CS(s)$ consists of m and $m + 1$.*

Proof. By Lemma 3.5, either $CS(s) = ((\ell, \ell))$ or $CS(s)$ consists of ℓ and $\ell + 1$ with $\ell \in \mathbb{Z}_+$. By Hypothesis B together with Remark 4.1(1), $\phi(\alpha)$ involves a subword w_i whose S -sequence is $(m + 1)$, so $CS(\phi(\alpha)) = CS(u_s) = CS(s)$ must contain a term of the form $m + 1 + c$, where $c \in \mathbb{Z}_+ \cup \{0\}$. First consider the case where $CS(s) = ((\ell, \ell))$. Since $CS(s)$ contains a term $m + 1 + c$, we have $\ell \geq m + 1$. Here, by Lemma 4.5(1), ℓ is not equal to $m + 1$. By Lemma 4.5(2), ℓ is not equal to $m + 1 + d$ for any $d \in \mathbb{Z}_+$ except $2m$. However, Lemma 4.5(3) implies that ℓ is not equal to $2m$, so that there remains no possibility for ℓ . Next consider the case where $CS(s)$ consists of ℓ and $\ell + 1$. By Lemma 4.5(2), none of ℓ and $\ell + 1$ is equal to $m + 1 + d$ for any $d \in \mathbb{Z}_+$ except $2m$. So $\ell \leq m$. On the other hand, since $CS(s)$ contains a term $m + 1 + c$, we have $\ell + 1 \geq m + 1$. Therefore $\ell = m$. \square

Next, we study the case where $r = [m, 1, n]$ with $m, n \geq 3$. Recall from Remark 4.1(2) that $CS(r) = ((n\langle m + 1 \rangle, m, n\langle m + 1 \rangle, m))$, where $S_1 = (n\langle m + 1 \rangle)$ and $S_2 = (m)$. Recall also $S(w_{i,b}) = S(w_{i,e}) = (m + 1)$ and $S(w'_{i,b}) = S(w'_{i,e}) = (m + 1)$ for every i .

Lemma 4.7. *Let $r = [m, 1, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, $CS(s)$ contains $m + 1$ as a term.*

Proof. The assertion immediately follows from the fact that $\phi(\alpha) = CS(u_s) = CS(s)$ involves a subword w_i whose S -sequence is $(n\langle m + 1 \rangle)$ with $n \geq 3$. \square

The following lemma is a counterpart of Lemma 4.4.

Lemma 4.8. *Let $r = [m, 1, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, the following hold for every i .*

- (1) $S(z_i y_{i+1}) \neq (m + 2)$.
- (2) $S(w_{i,e} z_i y_{i+1}) \neq (m + 2)$ and $S(z_i y_{i+1} w_{i+1,b}) \neq (m + 2)$.

Proof. The proofs of (1) and (2) are parallel to those of Lemma 4.4(1) and (2), respectively. \square

Lemma 4.9. *Let $r = [m, 1, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, no term of $CS(s)$ can be of the form $m + 2$.*

Proof. Suppose on the contrary that $CS(s)$ contains a term of the form $m + 2$. Let v be a subword of the cyclic word (u_s) corresponding to a term $m + 2$. Without loss of generality, we may assume that

- (i) $v = z_0 y_1$ with $|z_0|, |y_1| \neq 0$;
- (ii) $v = z_0 w_{1,b}$ with $|z_0| \neq 0$; or
- (iii) $v = w_{1,e} y_2$ with $|y_2| \neq 0$.

However, (i) is impossible by Lemma 4.8(1), and (ii) and (iii) are impossible by Lemma 4.8(2). \square

Corollary 4.10. *Let $r = [m, 1, n]$, where $m, n \geq 3$ are integers. Under Hypothesis B, either $CS(s) = ((m + 1, m + 1))$ or $CS(s)$ consists of m and $m + 1$.*

Proof. By Lemma 4.7, $CS(s)$ contains a term $m + 1$. Also by Lemma 4.9, $CS(s)$ does not contain a term $m + 2$. Hence, by Lemma 3.5, we obtain the desired result. \square

4.2. The case when Hypothesis C holds. We next assume that Hypothesis C holds. We also begin with the following remark before introducing two technical lemmas concerning the sequence $S(z_i y_{i+1})$ accordingly as $r = [m, n]$ and $r = [m, 1, n]$.

Remark 4.11. (1) If $r = [m, n]$, where $m, n \geq 3$ are integers, then $CS(r) = ((m + 1, (n - 1)\langle m \rangle, m + 1, (n - 1)\langle m \rangle))$, where $S_1 = (m + 1)$ and $S_2 = ((n - 1)\langle m \rangle)$. So, in Hypothesis C, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell_1, (n - 1)\langle m \rangle, \ell_2)$, where $1 \leq \ell_1, \ell_2 \leq m$ are integers.

(2) If $r = [m, 1, n]$, where $m, n \geq 3$ are integers, then $CS(r) = ((n\langle m + 1 \rangle, m, n\langle m + 1 \rangle, m))$, where $S_1 = (n\langle m + 1 \rangle)$ and $S_2 = (m)$. So, in Hypothesis C, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell_1, n_1\langle m + 1 \rangle, \ell_2, m, n_2\langle m + 1 \rangle, \ell_2)$, where $0 \leq \ell_1, \ell_2 \leq m$ and $0 \leq n_1, n_2 \leq n - 1$ are integers

such that a pair (ℓ_j, n_j) cannot be $(0, 0)$ for $j = 1, 2$. In particular, $S(y_{i,b}) = (\ell)$ with $1 \leq \ell \leq m + 1$. The same is true for $S(z_{i,e})$, $S(y'_{i,b})$ and $S(z'_{i,e})$.

Lemma 4.12. *Let $r = [m, n]$, where $m, n \geq 3$ are integers. Under Hypothesis C, the following hold for every i .*

- (1) $S(z_i y_{i+1})$ is not equal to $(m - 1)$ nor (m) .
- (2) $S(z_i y_{i+1})$ is not equal to $(m - 1, m)$, $(m, m - 1)$ nor (m, m) .

Proof. (1) Suppose on the contrary that $S(z_1 y_2) = (m)$. (The other case is treated analogously.) Since $|z_1|, |y_2| > 0$, we have $|z_1| = d$ and $|y_2| = m - d$ for some $1 \leq d \leq m - 1$. Here, if $J = M$ (see Figure 7(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term m and a term $m + 2$, contradicting Lemma 3.5. On the other hand, let $J \subsetneq M$. Since none of $S(\phi(e'_j))$ contains S_2 in its interior (see Lemma 3.20(2)), we see that the initial vertex of e'_2 lies in the (central) segment of ∂D_1^- corresponding to $S_2 = ((n - 1)\langle m \rangle)$ and that the terminal vertex of e'_3 lies in the (central) segment of ∂D_2^- corresponding to $S_2 = ((n - 1)\langle m \rangle)$. Thus we see that $CS(\phi(\partial D'_1)) = CS(r)$ contains a term of the form $m + 2 + c$ with $c \in \mathbb{Z}_+ \cup \{0\}$, as illustrated in Figure 7(b), a contradiction.

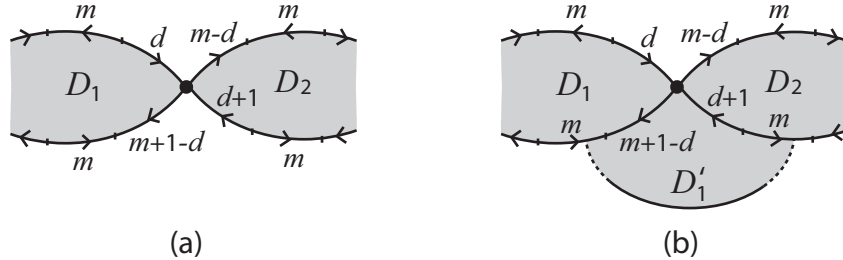


FIGURE 7. Lemma 4.12(1) where $S(z_1 y_2) = (d + (m - d))$.

(2) Suppose on the contrary that $S(z_1 y_2) = (m - 1, m)$. (The other cases are treated similarly.) Then $|z_1| = m - 1$ and $|y_2| = m$. Here, if $J = M$ (see Figure 8(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 1 and a term m , contradicting Lemma 3.5. On the other hand, if $J \subsetneq M$ (see Figure 8(b)), then $|\phi(e'_2)| = 2$ and $|\phi(e'_3)| = 1$, for otherwise we see, by using Lemma 3.20(2) as in the proof of (1), that a subsequence of the form $(\ell_1, 1, \ell_2)$ or of the form $(\ell_1, 2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ would occur in $S(\phi(e'_2 e'_3))$ which implies that $CS(\phi(\partial D'_1)) = CS(r)$ would contain a term 1 or a term 2, a contradiction.

Assuming that $e'_2, e'_3, e_3''^{-1}, e_2''^{-1}$ is a boundary cycle of D'_1 , this implies that $\phi(e_2''e_3'')$ contains a subword w such that $S(w)$ contains $(S_1, S_2) = (m+1, (n-1)\langle m \rangle)$ as a proper initial subsequence or $(S_2, S_1) = ((n-1)\langle m \rangle, m+1)$ as a proper terminal subsequence. But this is impossible by Corollary 3.17(2). \square

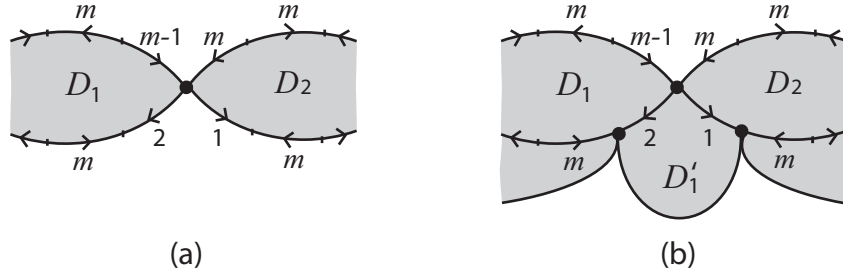


FIGURE 8. Lemma 4.12(2) where $S(z_1y_2) = (m-1, m)$.

Lemma 4.13. *Let $r = [m, 1, n]$, where $m, n \geq 3$ are integers. Under Hypothesis C, the following hold for every i .*

- (1) $S(z_{i,e}y_{i+1,b})$ is not equal to $(m-1)$ nor (m) .
- (2) $S(z_{i,e}y_{i+1,b})$ is not equal to $(m-1, m)$, $(m, m-1)$ nor (m, m) .
- (3) $S(z_{i,e}y_{i+1,b})$ is not equal to $(m-1, m-1)$.
- (4) $S(z_{i,e}y_{i+1,b})$ is not equal to $(m, m+1)$ nor $(m+1, m)$.

Proof. The proofs of (1) and (2) are parallel to those of Lemma 4.12(1) and (2), respectively.

(3) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m-1, m-1)$. Then $|z_{1,e}| = |y_{2,b}| = m-1$. Moreover, we must have $z_1 = z_{1,e}$ and $y_2 = y_{2,b}$. To see this, suppose $z_1 \neq z_{1,e}$. (The other case is treated similarly.) Then, since $S_1 = (n\langle m+1 \rangle)$, we see that $S(w_1z_1)$ is of the form $(S_2, m+1, *)$, where $*$ is nonempty. So, $CS(\phi(\alpha)) = CS(s)$ contains a term $m+1$. This is a contradiction to Lemma 3.5, because $CS(\phi(\alpha)) = CS(s)$ also contains $m-1$ by the assumption.

Here, if $J = M$ (see Figure 9(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ involves both a term 2 and a term $m+1$. Since $m+1 \geq 4$, we obtain a contradiction to Lemma 3.5. On the other hand, if $J \subsetneq M$, then by Lemma 3.20(2) the initial vertex of e'_2 lies in the segment of ∂D_1^- corresponding to $S_2 = (m)$ and that the terminal vertex of e'_3 lies in the segment of ∂D_2^- corresponding to $S_2 = (m)$. This implies that a subsequence of the form $(\ell_1, 2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs

in $S(\phi(e'_2 e'_3))$ (see Figure 9(b)), so in $CS(\phi(\partial D'_1)) = CS(r)$ contains a term 2, a contradiction.

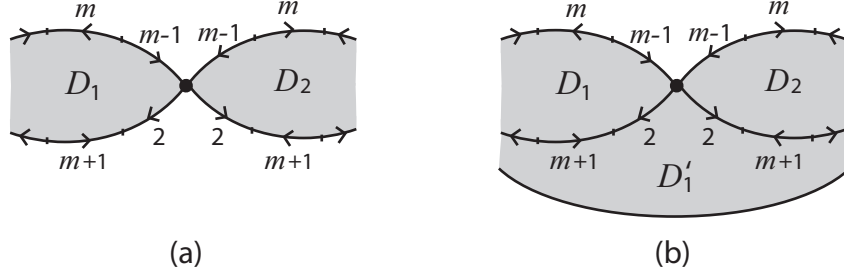


FIGURE 9. Lemma 4.13(3) where $S(z_{1,e}y_{2,b}) = (m-1, m-1)$.

(4) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m, m+1)$. (The other case is treated analogously.) Then $|z_{1,e}| = m$ and $|y_{2,b}| = m+1$. Here, if $J = M$ (see Figure 10), then $CS(\phi(\delta^{-1})) = CS(s')$ involves both a term m and a term $m+2$, contradicting Lemma 3.5. On the other hand, if $J \subsetneq M$, then we see, by using Lemma 3.20(2) as in the proof of (3), that a term of the form $m+2+c$ with $c \in \mathbb{Z}_+ \cup \{0\}$ occurs in $S(\phi(e'_2 e'_3))$, so in $CS(\phi(\partial D'_1)) = CS(r)$ contains a term $m+2+c$, a contradiction. \square

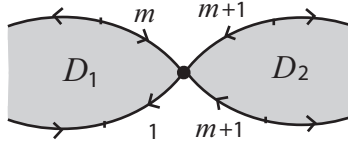


FIGURE 10. Lemma 4.13(4) where $S(z_{1,e}y_{2,b}) = (m, m+1)$.

5. PROOF OF MAIN THEOREM 2.5 FOR $r = [m, n]$ WITH $m, n \geq 3$

Suppose $r = [m, n]$, where $m, n \geq 3$ are integers. For two distinct elements $s, s' \in I_1(r) \cup I_2(r)$, suppose on the contrary that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely suppose that Hypothesis A holds. We will derive a contradiction in each case to consider. By Lemma 3.22, there are two big cases to consider.

Case 1. *Hypothesis B holds.*

By Corollary 4.6, $CS(s)$ consists of m and $m+1$. Without loss of generality, we may assume that a term m occurs in $S(z_1y_2)$. There are three possibilities:

- (i) $S(z_1y_2)$ consists of only m , where $S(z_1) = (n_1\langle m \rangle)$, $S(y_2) = (n_2\langle m \rangle)$, and $S(z_1y_2) = ((n_1 + n_2)\langle m \rangle)$ with $n_1, n_2 \in \mathbb{Z}_+ \cup \{0\}$;
- (ii) $S(z_1y_2)$ consists of only m , where $S(z_1) = (n_1\langle m \rangle, d)$, $S(y_2) = (m - d, n_2\langle m \rangle)$, and $S(z_1y_2) = ((n_1 + n_2 + 1)\langle m \rangle)$ with $n_1, n_2 \in \mathbb{Z}_+ \cup \{0\}$ and $d \in \{1, 2, \dots, m-1\}$;
- (iii) $S(z_1y_2)$ consists of m and $m+1$.

First assume that (i) occurs. Then $S(z'_1y'_2) = ((n'_1 + n'_2)\langle m \rangle)$ where $n'_1 = (n-1) - n_1$ and $n'_2 = (n-1) - n_2$. So $n'_1 + n'_2 = 2(n-1) - (n_1 + n_2)$ and hence either $S(z_1y_2)$ or $S(z'_1y'_2)$ contains $n-1$ consecutive m 's. If $J = M$, then this implies that either $s \notin I_1(r) \cup I_2(r)$ or $s' \notin I_1(r) \cup I_2(r)$ by Proposition 3.12, contradicting the hypothesis of the theorem. On the other hand, if $J \subsetneq M$, then the above observation implies that either $S(z_1y_2)$ contains $n-1$ consecutive m 's and so $s \notin I_1(r) \cup I_2(r)$, or otherwise $S(z'_1y'_2)$ contains n consecutive m 's. The former case is impossible by the assumption. In the latter case, we see, by an argument using Lemma 3.20(1) as in the last step of the proof of Lemma 4.4(1), that a subsequence of the form $(\ell_1, n\langle m \rangle, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(e'_2e'_3))$, so in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction. Next assume that (ii) occurs. Then $S(z'_1y'_2) = ((n'_1 + n'_2 + 1)\langle m \rangle)$, where $n'_1 = (n-2) - n_1$ and $n'_2 = (n-2) - n_2$. By using the identity $n'_1 + n'_2 + 1 = 2(n-1) - (n_1 + n_2 + 1)$, this case is treated as in the case when (i) occurs. Finally assume that (iii) occurs. Then we must have $S(z_{1,e}y_{2,b}) = (m+1)$, but this is a contradiction to Lemma 4.4(1).

Case 2. *Hypothesis C holds.*

By Remark 4.11(1), $CS(s)$ contains $((n-1)\langle m \rangle)$ as a proper subsequence. Thus Lemma 3.5 implies that $CS(s)$ consists of $\{m-1, m\}$ or $\{m, m+1\}$. Moreover, since $n \geq 3$, this together with Lemma 3.5 implies that $CS(s)$ does not contain $(m-1, m-1)$ nor $(m+1, m+1)$ as a subsequence.

Case 2.a. *$CS(s)$ consists of $m-1$ and m .*

In this case, a term $m-1$ should occur in $S(z_iy_{i+1})$ for some i . Since $CS(s)$ does not contain $(m-1, m-1)$ and since each of $S(z_i)$ and $S(y_i)$ has length 1, we see that $S(z_iy_{i+1})$ is equal to $(m-1)$, $(m, m-1)$ or $(m-1, m)$. But, this is impossible by Lemma 4.12.

Case 2.b. *$CS(s)$ consists of m and $m+1$.*

In this case, $CS(s)$ contains both $S_1 = (m + 1)$ and $S_2 = ((n - 1)\langle m \rangle)$ as subsequences. Hence, by Proposition 3.12, $s \notin I_1(r) \cup I_2(r)$, contradicting the hypothesis of the theorem. \square

6. PROOF OF MAIN THEOREM 2.5 FOR $r = [m, 1, n]$ WITH $m, n \geq 3$

Suppose $r = [m, 1, n]$, where $m, n \geq 3$ are integers. For two distinct elements $s, s' \in I_1(r) \cup I_2(r)$, suppose on the contrary that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely suppose that Hypothesis A holds. We will derive a contradiction in each case to consider. By Lemma 3.22, there are two big cases to consider.

Case 1. *Hypothesis B holds.*

By Corollary 4.10, we have the following two subcases.

Case 1.a. $CS(s) = ((m + 1, m + 1))$.

Since $\phi(\alpha)$ involves a subword w_i whose S -sequence is $(n\langle m + 1 \rangle)$ and since $n \geq 3$, this is impossible.

Case 1.b. $CS(s)$ consists of m and $m + 1$.

In this case, we see that $CS(s)$ contains a subsequence $S_1 = (n\langle m + 1 \rangle)$. Since it also contains a subsequence $S_2 = (m)$ by the assumption, we see by Proposition 3.12 that $s \notin I_1(r) \cup I_2(r)$, contradicting the hypothesis of the theorem.

Case 2. *Hypothesis C holds.*

By Remark 4.11(2), $CS(s)$ contains the term m . So, by Lemma 3.5, Case 2 is reduced to the following three subcases: $CS(s) = ((m, m))$, $CS(s)$ consists of $\{m - 1, m\}$, or $CS(s)$ consists of $\{m, m + 1\}$.

Case 2.a. $CS(s) = ((m, m))$.

In this case, there is only one possibility: J consists of one 2-cell, namely $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+)) = ((S(z_1 y_1), S(w_1))) = ((m, m))$. Then $S(z_1 y_1) = S(z_{1,e} y_{1,b}) = (m)$, contradicting Lemma 4.13(1).

Case 2.b. $CS(s)$ consists of $m - 1$ and m .

In this case, a term $m - 1$ must occur in $S(z_{i,e} y_{i+1,b})$ for some i . But by Lemma 4.13(1), (2) and (3), this is impossible.

Case 2.c. $CS(s)$ consists of m and $m + 1$.

Without loss of generality, we may assume that $m + 1$ occurs in $S(z_1 y_2)$. There are three possibilities:

- (i) $S(z_1 y_2)$ consists of only $m + 1$, where $S(z_1) = (n_1 \langle m + 1 \rangle)$, $S(y_2) = (n_2 \langle m + 1 \rangle)$, and $S(z_1 y_2) = ((n_1 + n_2) \langle m + 1 \rangle)$ with $n_1, n_2 \in \mathbb{Z}_+ \cup \{0\}$;
- (ii) $S(z_1 y_2)$ consists of only $m + 1$, where $S(z_1) = (n_1 \langle m + 1 \rangle, d)$, $S(y_2) = (m + 1 - d, n_2 \langle m + 1 \rangle)$, and $S(z_1 y_2) = ((n_1 + n_2 + 1) \langle m + 1 \rangle)$ with $n_1, n_2 \in \mathbb{Z}_+ \cup \{0\}$ and $d \in \{1, \dots, m\}$;
- (iii) $S(z_1 y_2)$ consists of m and $m + 1$.

However, by an argument as in Case 1(i) and (ii) in Section 5, we can see that neither (i) nor (ii) can happen. (In the above, we need to appeal to Lemma 3.20(2) instead of Lemma 3.20(1).) So only (iii) can occur. But then a term m must occur in $S(z_{1,e} y_{2,b})$, a contradiction to Lemma 4.13(1), (2) and (4). \square

7. PRELIMINARY RESULTS FOR THE GENERAL CASES

The remainder of this paper is devoted to the proof of Main Theorem 2.5 when r is *general*, namely either $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$, or $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. In the remainder of this paper, we assume that r is general.

Remark 7.1. (1) Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Then $\tilde{r} = [m_2 - 1, m_3, \dots, m_k]$ by Lemma 3.7, and so, by Proposition 3.8, $CS(\tilde{r}) = ((T_1, T_2, T_1, T_2))$, where T_1 begins and ends with m_2 and T_2 begins and ends with $m_2 - 1$. Thus by Lemma 3.11(4), we obtain that $CS(r) = ((S_1, S_2, S_1, S_2))$, where S_1 begins and ends with $(m + 1, (m_2 - 1) \langle m \rangle, m + 1)$, and S_2 begins and ends with $(m_2 \langle m \rangle)$.

(2) Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Then $\tilde{r} = [m_3, \dots, m_k]$ by Lemma 3.7, and so, by Proposition 3.8, $CS(\tilde{r}) = ((T_1, T_2, T_1, T_2))$, where T_1 begins and ends with $m_3 + 1$ and T_2 begins and ends with m_3 . Thus by Lemma 3.11(2), we obtain that $CS(r) = ((S_1, S_2, S_1, S_2))$, where S_1 begins and ends with $((m_3 + 1) \langle m + 1 \rangle)$, and S_2 begins and ends with $(m, m_3 \langle m + 1 \rangle, m)$.

The aim of this section is to prove the following proposition.

Proposition 7.2. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$, or $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Then under Hypothesis A, both $CS(s)$ and $CS(s')$ consist of m and $m + 1$. Moreover $CS(s)$ contains S_1 or S_2 as a subsequence accordingly as Hypothesis B or Hypothesis C is satisfied.*

In the following, we prove the proposition only for $CS(s)$. By applying the same argument to the annular diagram reversing the outer and inner boundaries, we see that the assertion also holds for $CS(s')$.

7.1. The case when Hypothesis B holds. In this subsection, we study the case when Hypothesis B holds.

Suppose $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Then S_1 begins and ends with $(m+1, (m_2-1)\langle m \rangle, m+1)$ by Remark 7.1(1). Thus Hypothesis B implies that $CS(\phi(\alpha)) = CS(s)$ contains a term m and a term $m+1+\ell$ for some $\ell \geq 0$. Hence $CS(s)$ must consist of m and $m+1$ by Lemma 3.5. Moreover, since S_1 begins and ends with $m+1$, this observation together with Hypothesis B implies that $CS(s)$ contains S_1 as a subsequence. Thus Proposition 7.2 holds in this case.

Suppose $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. The following lemma is needed for the proof of Lemma 7.5, by which we prove Proposition 7.2 for this type of r .

Lemma 7.3. *Let $r = [2, 1, m_3, \dots, m_k]$, where $k \geq 4$. Under Hypothesis B, the following hold.*

- (1) $S(z_{i,e}y_{i+1,b}) \neq (4)$.
- (2) $S(w_{i,e}z_iy_{i+1,b}) \neq (4)$ and $S(z_{i-1,e}y_iw_{i,b}) \neq (4)$.
- (3) $S(w_{i,e}z_iy_{i+1,b}) \neq (5)$ and $S(z_{i-1,e}y_iw_{i,b}) \neq (5)$.
- (4) $S(w_{i,e}z_iy_{i+1}w_{i+1,b}) \neq (6)$.

Proof. (1) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (4)$. Then we must have that $z_1 = z_{1,e}$ and $y_2 = y_{2,b}$. To see this, suppose that $z_1 \neq z_{1,e}$. (The other case is treated similarly.) Then, since S_2 begins with $(2, m_3\langle 3 \rangle, 2)$, we see that $S(w_1z_1)$ is of the form $(S_1, 2, *)$, where $*$ is nonempty. So, $CS(\phi(\alpha)) = CS(s)$ contains a term 2. This is a contradiction to Lemma 3.5, because $CS(\phi(\alpha))$ also contains a term 4 by the assumption. Hence, $S(z_1) = S(z_{1,e}) = (2)$ and $S(y_2) = S(y_{2,b}) = (2)$. Since S_2 begins and ends with $(2, m_3\langle 3 \rangle, 2)$, $S(z'_1y'_2)$ contains a subsequence of the form $(\ell_1, 6, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ (see Figure 11(a)). So if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 2 and a term 6, contradicting Lemma 3.5. Suppose $J \subsetneq M$. Then the above fact together with Lemma 3.20(1) implies that $S(\phi(e'_2e'_3))$ contains the subsequence $(\ell_1, 6, \ell_2)$ (see Figure 11(b)). Thus, a term 6 occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction.

(2) Suppose on the contrary that $S(w_{1,e}z_1y_{2,b}) = (4)$. (The other case is treated similarly.) Then $|z_1| = 0$ and $|y_{2,b}| = 1$, because $|w_{1,e}| = 3$. Furthermore $y_2 = y_{2,b}$ as in the proof of (1). By using the fact that S_2 begins and

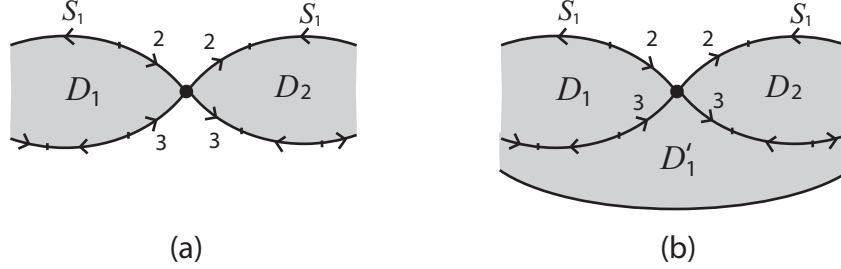


FIGURE 11. Lemma 7.3(1) where $S(z_{1,e}y_{2,b}) = (2 + 2)$.

ends with $(2, m_3\langle 3 \rangle, 2)$ and Lemma 3.20(1), we see that $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, 2, 1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ (see Figure 12(a)). So, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 1 and a term 3, contradicting Lemma 3.5. On the other hand, if $J \subsetneq M$, then a term 1 occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction.

(3) Suppose on the contrary that $S(w_{1,e}z_1y_{2,b}) = (5)$. (The other case is treated similarly.) Then $|z_1| = 0$ and $|y_{2,b}| = 2$. Furthermore $y_2 = y_{2,b}$ as in the proof of (1). So $S(y_2w_2)$ begins with a subsequence $(2, (m_3 + 1)\langle 3 \rangle)$, which implies that $CS(\phi(\alpha)) = CS(s)$ has a term 3. Since $CS(s)$ has a term 5 by assumption, this is a contradiction to Lemma 3.5.

(4) Suppose on the contrary that $S(w_{1,e}z_1y_2w_{2,b}) = (6)$. Then $|z_1| = |y_2| = 0$ and $S(w_{1,e}w_{2,b}) = (6)$. By using the fact that S_2 begins and ends with $(2, m_3\langle 3 \rangle, 2)$ and Lemma 3.20(1), we see that $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, 4, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ (see Figure 12(b)). So, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 2 and a term 4, contradicting Lemma 3.5. On the other hand, if $J \subsetneq M$, then a term 4 occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction. \square

Lemma 7.4. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 3$ and $k \geq 4$. Under Hypothesis B, the following hold.*

- (1) $S(z_{i,e}y_{i+1,b}) \neq (m + d)$ for any $d \in \mathbb{Z}_+$ with $2 \leq d \leq m$.
- (2) $S(w_{i,e}z_iy_{i+1,b}) \neq (m + 1 + d)$ and $S(z_{i-1,e}y_iw_{i,b}) \neq (m + 1 + d)$ for any $d \in \mathbb{Z}_+$ with $1 \leq d \leq m$.
- (3) $S(w_{i,e}z_iy_{i+1}w_{i+1,b}) \neq (2m + 2)$.

Proof. (1) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m + d)$ with $2 \leq d \leq m$. As in the proof of Lemma 7.3(1), we must have $z_1 = z_{1,e}$ and $y_2 = y_{2,b}$, for otherwise $CS(\phi(\alpha)) = CS(s)$ would contain both a term m and a term $m + d$

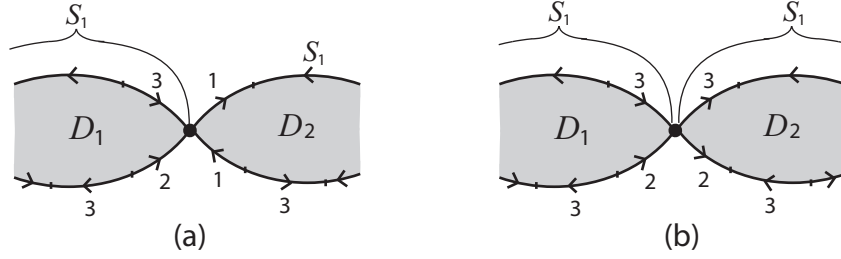


FIGURE 12. (a) Lemma 7.3(2) where $S(w_{1,e}z_1y_{2,b}) = (3+0+1)$, and (b) Lemma 7.3(4) where $S(w_{1,e}z_1y_2w_{2,b}) = (3+0+3)$.

contradicting Lemma 3.5. So if $2 \leq d \leq m-1$, then the proof is parallel to that of Lemma 4.4(1). Now let $d = m$. Then $|z_1| = |y_2| = m$, and so $CS(\phi(\alpha)) = CS(s)$ has a term $2m$. Moreover, $S(y_2w_2)$ begins with $(m, (m_3 + 1)\langle m + 1 \rangle)$, and hence $CS(s)$ has a term $m + 1$, as shown in Figure 13. But since $m \geq 3$, we have $2m > m + 2$. This gives a contradiction to Lemma 3.5.

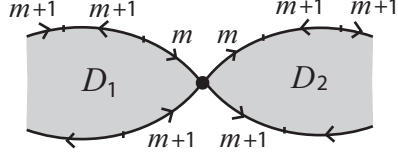


FIGURE 13. Lemma 7.4(1) where $S(z_{1,e}y_{2,b}) = (m + m)$.

(2) Suppose on the contrary that $S(w_{1,e}z_1y_{2,b}) = (m+1+d)$. (The other case is treated analogously.) Then $|z_1| = 0$ and $|y_{2,b}| = d$. Furthermore $y_2 = y_{2,b}$ as in the proof of (1). So if $1 \leq d \leq m-1$, then the proof is parallel to that of Lemma 4.4(2). Now let $d = m$. Then $|y_2| = m$ and $CS(s)$ has a term $2m + 1$. Also, as shown in the proof of (1), $CS(s)$ has a term $m + 1$. But since $m \geq 3$, we have $2m + 1 > m + 2$. This gives a contradiction to Lemma 3.5.

(3) Suppose on the contrary that $S(w_{1,e}z_1y_2w_{2,b}) = (2m + 2)$. Then $|z_1| = |y_2| = 0$ and $S(w_{1,e}w_{2,b}) = (2m + 2)$. Assume first that $J = M$. Then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term $m + 1$ and a term $2m$ as illustrated in Figure 14(a). Since $m \geq 3$, this gives a contradiction to Lemma 3.5. Assume next that $J \subsetneq M$. Then by Lemma 3.20(1), none of $S(\phi(e'_1))$ and $S(\phi(e'_2))$ contains a subsequence S_1 which begins and ends with $((m_3 + 1)\langle m + 1 \rangle)$. This means that the initial vertex of e'_2 lies in the interior of the segment

of ∂D_1^- corresponding to S_1 . Similarly, the terminal vertex of e'_3 lies in the interior of the segment of ∂D_2^- corresponding to S_1 . Hence, we see from Figure 14(b) that $S(\phi(e'_2 e'_3))$ contains a subsequence of the form $(\ell_1, 2m, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. This implies that a term $2m$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, which is a contradiction. \square

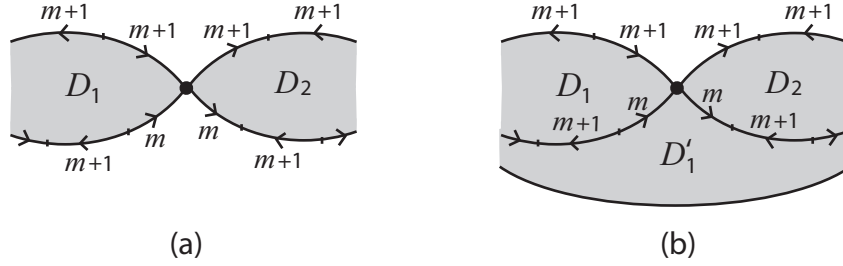


FIGURE 14. Lemma 7.4(3) where $S(w_{1,e}z_1y_2w_{2,b}) = ((m+1) + 0 + 0 + (m+1))$.

Lemma 7.5. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Under Hypothesis B, no term of $CS(s)$ can be of the form $m+1+d$ with $d \in \mathbb{Z}_+$.*

Proof. Suppose on the contrary that $CS(s)$ contains a term $m+1+d$. Let v be a subword of the cyclic word (u_s) corresponding to a term $m+1+d$. Without loss of generality, we may assume that

- (i) v contains $z_{1,e}y_{2,b}$ with $|z_{1,e}|, |y_{2,b}| \neq 0$;
- (ii) v contains $w_{1,e}y_{2,b}$ with $|y_{2,b}| \neq 0$;
- (iii) v contains $z_{0,e}w_{1,e}$ with $|z_{0,e}| \neq 0$; or
- (iv) v contains $w_{1,e}w_{2,b}$ with $|z_1| = |y_2| = 0$.

However, (i) is impossible by Lemma 7.3(1) or Lemma 7.4(1) accordingly as $m = 2$ or $m \geq 3$. Also (ii) and (iii) are impossible by Lemma 7.3(2)–(3) or Lemma 7.4(2) accordingly as $m = 2$ or $m \geq 3$. Finally (iv) is impossible by Lemma 7.3(4) or Lemma 7.4(3) accordingly as $m = 2$ or $m \geq 3$. \square

Corollary 7.6. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Under Hypothesis B, the conclusion of Proposition 7.2 holds.*

Proof. By Lemma 3.5, either $CS(s) = ((\ell, \ell))$ or $CS(s)$ consists of ℓ and $\ell+1$, for some $\ell \in \mathbb{Z}_+$. Since $CS(\phi(\alpha)) = CS(s)$ must contain a term of the form $m+1+c$ with $c \in \mathbb{Z}_+ \cup \{0\}$, we have $\ell \geq m+1$ in the first case and $\ell \geq m$ in the second case. First, if $CS(s) = ((\ell, \ell))$, then $\ell = m+1$

by Lemma 7.5, namely $CS(s) = ((m+1, m+1))$. This happens only when J consists of only one 2-cell with $CS(\phi(\partial D_1^+)) = ((S_1)) = ((m+1, m+1))$. Then $CS(\phi(\partial D_1^-)) = ((S_2, S_1, S_2))$, and so $S(\phi(e'_2 e'_1))$ contains a subsequence of the form $(\ell_1, S_2, S_2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. So, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains two consecutive m 's and two consecutive $m+1$'s, contradicting Lemma 3.5. On the other hand, if $J \subsetneq M$, then a subsequence (S_2, S_2) occurs in $CS(\phi(\partial D_1')) = CS(r)$, a contradiction. Thus $CS(s) \neq ((\ell, \ell))$, and so $CS(s)$ consists of ℓ and $\ell+1$. By Lemma 7.5, we have $\ell+1 \leq m+1$, so that $\ell = m$, as desired. Hence, $CS(s)$ consists of m and $m+1$. As already observed in the beginning of this subsection, this implies that $CS(s)$ contains S_1 as a subsequence. \square

7.2. The case when Hypothesis C holds. We next assume that Hypothesis C holds. In this case, $CS(s)$ contains S_2 as a subsequence.

Suppose $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Then S_2 begins and ends with $(m, m_3 \langle m+1 \rangle, m)$ by Remark 7.1(2). Hence $CS(\phi(\alpha)) = CS(s)$ contains terms m and $m+1$, and therefore $CS(\phi(\alpha)) = CS(s)$ consists of m and $m+1$ by Lemma 3.5. So, Proposition 7.2 holds in this case.

On the other hand, for $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$, we prove the following lemmas, by which we prove Proposition 7.2 for this type of r .

Lemma 7.7. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Under Hypothesis C, $CS(s)$ consists of at least three terms including m .*

Proof. The assertion immediately follows from the fact that $CS(\phi(\alpha)) = CS(s)$ properly contains S_2 which begins and ends with $(m_2 \langle m \rangle)$ (see Remark 7.1(1)). \square

Lemma 7.8. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Under Hypothesis C, the following hold for every i .*

- (1) $S(z_{i,e} y_{i+1,b}) \neq (m-1)$.
- (2) $S(z_{i,e} y_{i+1,b}) \neq (m-1, m-1)$.
- (3) $S(z_{i,e} y_{i+1,b}) \neq (m-1, m)$ and $S(z_{i,e} y_{i+1,b}) \neq (m, m-1)$.

Proof. (1) Suppose on the contrary that $S(z_{1,e} y_{2,b}) = (m-1)$. Then we have $z_1 = z_{1,e}$ and $y_2 = y_{2,b}$, for otherwise $CS(\phi(\alpha)) = CS(s)$ contains both a term $m-1$ and a term $m+1$, contradicting Lemma 3.5. By using Lemma 3.20(2) as in the proof of Lemma 4.12(1), we see that $S(\phi(e'_2 e'_3))$ contains a subsequence of the form $(\ell_1, m+3, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. So, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term m and a term $m+3$, contradicting Lemma 3.5. On

the other hand, if $J \subsetneq M$, then a term $m + 3$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction.

(2) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m - 1, m - 1)$. Then $CS(\phi(\alpha)) = CS(s)$ involves two consecutive $m - 1$'s. On the other hand, since $CS(s)$ contains S_2 , which begins and ends with $(m_2\langle m \rangle)$, we see that $CS(s)$ also contains two consecutive m 's. This is a contradiction to Lemma 3.5.

(3) Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m - 1, m)$. (The other case is treated similarly.) As in the proof of (1), $|z_{1,e}| = m - 1$, $|y_{2,b}| = m$, $z_1 = z_{1,e}$ and $y_2 = y_{2,b}$. By using Lemma 3.20(2) as in the proof of Lemma 4.12(2), we see that $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, 2, 1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. Hence, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 1 and a term $m + 1$. Since $m + 1 \geq 3$, we have a contradiction to Lemma 3.5. On the other hand, if $J \subsetneq M$, then a term 1 occurs in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction. \square

Corollary 7.9. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Under Hypothesis C, the conclusion of Proposition 7.2 holds.*

Proof. By Lemma 7.7, $CS(s)$ consists of at least three terms including m . Also, Lemma 7.8 shows that no term of $CS(s)$ can be $m - 1$. Hence by Lemma 3.5, $CS(s)$ must consist of m and $m + 1$. Moreover, we have already observed that $CS(s)$ contains S_2 as a subsequence. \square

Thus we have proved Proposition 7.2.

8. TRANSFORMATION OF DIAGRAMS FOR THE GENERAL CASES

We first introduce a concept for a vertex of M to be converging, diverging or mixing. To this end, we subdivide the edges of M so that the label of any oriented edge in the subdivision has length 1. We call each of the edges in the subdivision a *unit segment* in order to distinguish them from the edges in the original M .

Definition 8.1. (1) A vertex x in M is said to be *converging* (resp. *diverging*) if the set of labels of incoming unit segments of x is $\{a, b\}$ (resp. $\{a^{-1}, b^{-1}\}$). See Figure 15 and its caption for description.

(2) A vertex x in M is said to be *mixing* if the set of labels of incoming unit segments of x is $\{a, a^{-1}, b, b^{-1}\}$. See Figure 16 and its caption for description.

As we declared in the previous section, we assume that r is general. The main purpose of this section is to show that, under Hypothesis A, we may modify M so that every vertex x in M with $\deg_M(x) = 4$ is either converging or diverging (see Corollary 8.12).

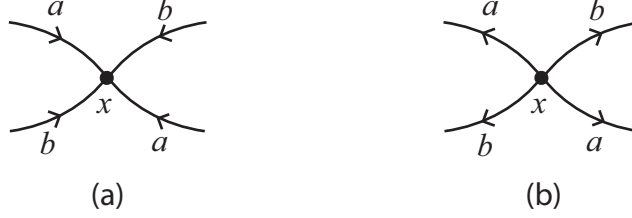


FIGURE 15. Orient each of the unit segment so that the associated label is equal to a or b . Then a vertex x is (a) converging (resp. (b) diverging) if all unit segments incident on x are oriented so that they are converging into x (resp. diverging from x).



FIGURE 16. A vertex x is mixing if it looks like as in the above when we orient the segments as in Convention 4.3, namely, the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words.

8.1. The case when vertices lie in the outer boundary layer. We first treat a vertex x in the outer boundary layer J with $\deg_J(x) = 4$. To do this, we need several lemmas.

Lemma 8.2. *Assume that r is general. Under Hypothesis B, suppose that the vertex between D_i and D_{i+1} is either converging or diverging. Then none of the following occurs.*

- (1) $S(\phi(\partial D_i^+))$ ends with S_1 .
- (2) $S(\phi(\partial D_{i+1}^+))$ begins with S_1 .
- (3) $S(\phi(\partial D_i^-))$ ends with S_1 .
- (4) $S(\phi(\partial D_{i+1}^-))$ begins with S_1 .

Proof. We may assume that $i = 1$ and that the vertex is diverging. Suppose on the contrary that (1) occurs, namely suppose that $S(\phi(\partial D_1^+))$ ends

with S_1 . Then $S(\phi(\partial D_1^-))$ ends with (S_1, S_2) . Since the vertex is diverging, $S(\phi(\partial D_1^-)\phi(\partial D_2^-))$ contains a subsequence (S_1, S_2, d) for some $d \in \mathbb{Z}_+$. Suppose $J = M$. Then $CS(\phi(\delta^{-1})) = CS(s')$ contains S_2 . (See Figure 17(a), keeping in mind Convention 4.3.) Moreover, the subsequence S_1 of $S(\phi(\partial D_1^-))$ also forms a subsequence of $CS(\phi(\delta^{-1})) = CS(s')$, because $CS(s')$ consists of m and $m+1$ (Proposition 7.2) and S_1 begins and ends with $m+1$. Thus $CS(s')$ contains both S_1 and S_2 , yielding that $s' \notin I_1(r) \cup I_2(r)$ by Proposition 3.12, a contradiction. Suppose $J \subsetneq M$. Then we see in the following that the two 2-cells D_1 and D'_1 in Figure 17(b) form a reducible pair, contradicting that M is a reduced diagram. To see this, let γ (resp. γ') be the boundary cycle of D_1 (resp. D'_1) which goes around the boundary in clockwise (resp. counter-clockwise) direction starting from the vertex between D_1 and D_2 . Let γ_0 be the common initial segment of γ and γ' such that $S(\phi(\gamma_0)) = S_2$. Then we see by using Lemma 3.20(1) that the terminal point of γ_0 (= the initial point of e'_2) is contained in the interior of the segment of ∂D_1^- corresponding to the segment S_1 (see Figure 17(b)). Thus, in both reduced words $\phi(\gamma)$ and $\phi(\gamma')$, there are “sign changes” just before and after $\phi(\gamma_0)$. Hence $S(\phi(\gamma_0)) = S_2$ is a subsequence of both $CS(\phi(\gamma)) = CS(r)$ and $CS(\phi(\gamma')) = CS(r)$. By Proposition 3.8(2) and the fact that $\phi(\gamma)$ and $\phi(\gamma')$ shares $\phi(\gamma_0)$ as a common initial word, we must have $\phi(\gamma) \equiv \phi(\gamma')$; so D_1 and D'_1 form a reducible pair. Thus we have proved that (1) does not occur. Similarly, we can prove that (2) does not occur.

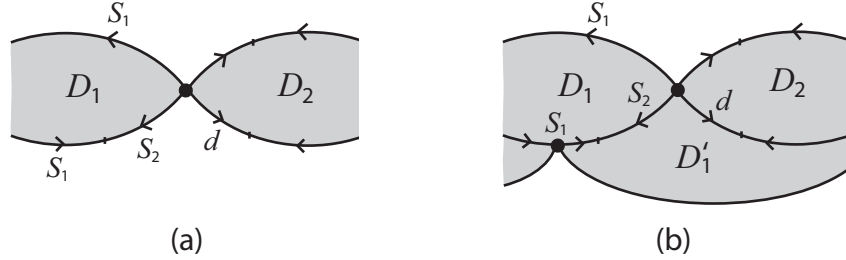


FIGURE 17. Lemma 8.2(1) where $S(\phi(\partial D_1^+))$ ends with S_1 .

Suppose on the contrary that (3) occurs, i.e., $S(\phi(\partial D_1^-))$ ends with S_1 . Then $S(\phi(\partial D_1^+))$ ends with (S_1, S_2) . Since the vertex is diverging, this subsequence S_2 of $S(\phi(\partial D_1^+))$ is also a subsequence of $CS(\phi(\alpha)) = CS(s)$. Moreover, we can see as in the proof of (1) that the subsequence S_1 of $S(\phi(\partial D_1^+))$ forms a

subsequence of $CS(\phi(\alpha))$. Thus $CS(\phi(\alpha))$ contains both S_1 and S_2 as subsequences and so $s \notin I_1(r) \cup I_2(r)$ by Proposition 3.12, a contradiction. The assertion (4) is proved similarly. \square

Lemma 8.3. *Assume that r is general. Under Hypothesis B, we may assume that the following hold for every face D_i of J .*

- (1) $S(\phi(\partial D_i^+))$ contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.
- (2) $S(\phi(\partial D_i^-))$ contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.

To be precise, we can modify the reduced annular diagram M into a reduced annular diagram M' keeping the outer and inner boundary labels unchanged so that every 2-cell of the outer boundary layer of M' satisfies the above conditions.

Proof. Suppose that the assertion does not hold. Then one of the four (prohibited) conditions in Lemma 8.2 holds. In particular, the vertex between D_i and D_{i+1} is not converging nor diverging.

Suppose that condition (1) in Lemma 8.2 occurs. Then we may assume that $S(\phi(\partial D_1^+))$ ends with S_1 and so $|z_1| = 0$. Then $S(z'_1) = S_2$, and so $S(z'_{1,e}) = (m)$. Hence the sequence $S(z'_{1,e}y'_2w'_2)$ begins with a subsequence of the form either (m, d) or $(m + d)$, where $d \in \mathbb{Z}_+$. Suppose that $S(z'_{1,e}y'_2w'_2)$ begins with a subsequence of the form (m, d) . Then the vertex between D_1 and D_2 is either converging or diverging, a contradiction. (In fact, since $CS(s) = CS(\phi(\alpha))$ consists of m and $m + 1$ by Proposition 7.2 and since $S(\phi(\partial D_1^+))$ ends with $m + 1$, there is a “sign change” between $\phi(\partial D_1^+)$ and $\phi(\partial D_2^+)$. Thus J is locally as illustrated in Figure 17(a) up to simultaneous change of the edge orientations.) So $S(z'_{1,e}y'_2w'_2)$ must begin with a subsequence of the form $(m + d)$. Since $S(\phi(\partial D_1^+))$ ends with a term $m + 1$ and since $CS(\phi(\alpha)) = CS(s)$ consists of m and $m + 1$, the only possibility is that $d = 1$ and $S(w_{1,e}y_{2,b}) = (m + 1, m)$. Then, as illustrated in Figure 18, we may transform M so that $S(\phi(\partial D_1^+))$ ends with (S_1, m) . To be precise, we cut J at the black vertex in the left figure in Figure 18 and then identify the two white vertices. The resulting diagram is illustrated in the right figure in Figure 18, where the black vertex is the image of the white vertices. It should be noted that the boundary labels of J are unchanged by this operation and so we can glue $M - J$ to this new J . This modification does not change the boundary labels of M and the new vertex of J is converging or diverging. Hence we see by Lemma 8.2 that none of conditions (1)–(4) occurs at the vertex between D_1 and D_2 in this new diagram. Thus we have shown that condition (1) in Lemma 8.2 may be

assumed not to occur. Similarly, we can show that condition (2) in Lemma 8.2 may be assumed not to occur.

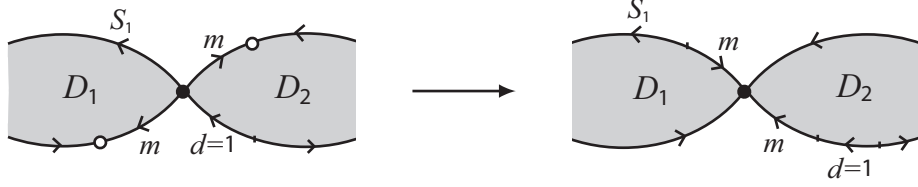


FIGURE 18. Lemma 8.3 where $S(z'_{1,e}y'_2w'_2)$ begins with $(m + d)$.

Suppose that condition (3) in Lemma 8.2 occurs. Then we may assume $S(\phi(\partial D_1^-))$ ends with S_1 and so $|z'_1| = 0$. Then $S(z_1) = S_2$, and so $S(z_{1,e}) = (m)$. Hence the sequence $S(z_{1,e}y_2w_2)$ begins with a subsequence of the form either (m, d) or $(m + d)$, where $d \in \mathbb{Z}_+$. If $S(z_{1,e}y_2w_2)$ begins with a subsequence of the form (m, d) , then we can see as in the previous case that the vertex between D_1 and D_2 is converging or diverging, a contradiction. So $S(z_{1,e}y_2w_2)$ must begin with a subsequence of the form $(m + d)$. Since $S(\phi(\partial D_1^+))$ ends with a term m and since $CS(\phi(\alpha)) = CS(s)$ consists of m and $m + 1$, the only possibility is that $d = 1$ and $S(w'_{1,e}y'_{2,b}) = (m + 1, m)$. Then, as illustrated in Figure 19, we may transform M so that $S(\phi(\partial D_1^-))$ ends with (S_1, m) . Since the new vertex is either converging or diverging, we see by Lemma 8.2 that none of conditions (1)–(4) occurs at the common vertex of D_1 and D_2 in this new diagram. Thus we have shown that condition (3) in Lemma 8.2 may be assumed not to occur. Similarly, we can show that condition (4) in Lemma 8.2 may be assumed not to occur. This completes the proof of Lemma 8.3. \square

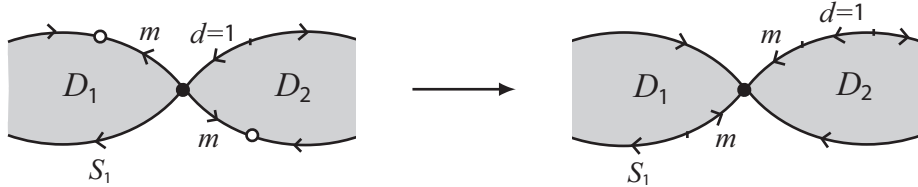


FIGURE 19. Lemma 8.3 where $S(z_{1,e}y_2w_2)$ begins with $(m + d)$.

In the remainder of this section, when we assume Hypothesis B, we always assume that the two conditions in Lemma 8.3 hold, namely the words y_i, z_i, y'_i

and z'_i which appear in the expressions of $\phi(\partial D_i^+)$ and $\phi(\partial D_i^-)$ are nonempty. Note that when we assume Hypothesis C, the same conditions always hold by the hypothesis.

Lemma 8.4. *Assume that r is general. Under Hypothesis B or Hypothesis C, the following hold for every i .*

- (1) If $S(z_{i,e}z'_{i,e}) = S(y'_{i+1,b}y_{i+1,b}) = (m)$, then $S(z_{i,e}y_{i+1,b}) \neq (m+1)$.
- (2) If $S(z_{i,e}z'_{i,e}) = S(y'_{i+1,b}y_{i+1,b}) = (m+1)$, then $S(z_{i,e}y_{i+1,b}) \neq (m)$.
- (3) If $S(z_{i,e}z'_{i,e}) = (m)$ and $S(y'_{i+1,b}y_{i+1,b}) = (m, m)$, then $S(z_{i,e}y_{i+1,b}) \neq (m+1)$.
- (4) If $S(z_{i,e}z'_{i,e}) = (m)$ and $S(y'_{i+1,b}y_{i+1,b}) = (m+1, m)$, then $S(z_{i,e}y_{i+1,b}) \neq (m+1)$.
- (5) If $S(z_{i,e}z'_{i,e}) = S(y'_{i+1,b}y_{i+1,b}) = (m+1)$, then $S(z_{i,e}y_{i+1,b}) \neq (m, m)$.
- (6) If $S(z_{i,e}z'_{i,e}) = (m, m+1)$ and $S(y'_{i+1,b}y_{i+1,b}) = (m+1)$, then $S(z_{i,e}y_{i+1,b}) \neq (m, m)$.
- (7) If $S(z_{i,e}z'_{i,e}) = (m+1, m+1)$ and $S(y'_{i+1,b}y_{i+1,b}) = (m+1)$, then $S(z_{i,e}y_{i+1,b}) \neq (m+1, m)$.

Proof. (1) Let $S(z_{1,e}z'_{1,e}) = S(y'_{2,b}y_{2,b}) = (m)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m+1)$ (see Figure 20(a)). Then $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, m-1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. Here, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains a term $m-1$, a contradiction to Proposition 7.2. On the other hand, if $J \subsetneq M$, then a term $m-1$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction (cf. Proof of Lemma 4.4(1) for the case $J \subsetneq M$).

(2) Let $S(z_{1,e}z'_{1,e}) = S(y'_{2,b}y_{2,b}) = (m+1)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m)$ (see Figure 20(b)). Then $S(\phi(e'_2e'_3))$ contains a term $m+2$. Here, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains a term $m+2$, a contradiction to Proposition 7.2. On the other hand, if $J \subsetneq M$, then a term of the form $m+2+c$ with $c \in \mathbb{Z}_+ \cup \{0\}$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction.

(3) Let $S(z_{1,e}z'_{1,e}) = (m)$ and $S(y'_{2,b}y_{2,b}) = (m, m)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m+1)$ (see Figure 21(a)). Then $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, m-1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. Here, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains a term $m-1$, a contradiction to Proposition 7.2. On the other hand, if $J \subsetneq M$, then a term $m-1$ occurs in $CS(\phi(\partial D'_1)) = CS(r)$, a contradiction.

(4) Let $S(z_{1,e}z'_{1,e}) = (m)$ and $S(y'_{2,b}y_{2,b}) = (m+1, m)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m+1)$ (see Figure 21(b)). Then $S(\phi(e'_2e'_3))$ contains

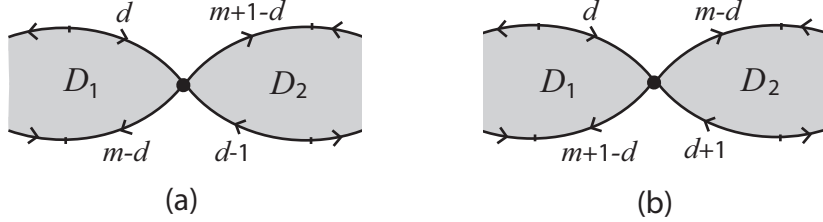


FIGURE 20. (a) Lemma 8.4(1), and (b) Lemma 8.4(2)

a subsequence of the form $(\ell_1, m-1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$. So, arguing as in the proof of (3), we reach a contradiction.

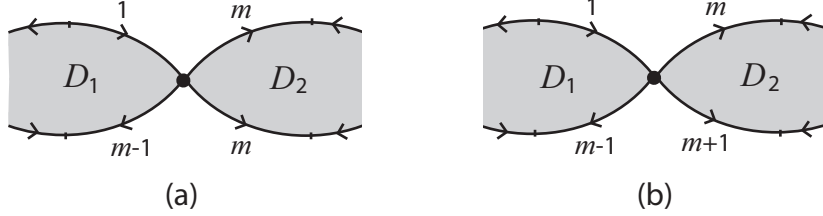


FIGURE 21. (a) Lemma 8.4(3), and (b) Lemma 8.4(4)

(5) Let $S(z_{1,e}z'_{1,e}{}^{-1}) = S(y'_{2,b}{}^{-1}y_{2,b}) = (m+1)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m, m)$ (see Figure 22(a)). Here, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains a term 1, a contradiction to Proposition 7.2. On the other hand, if $J \subsetneq M$, then $S(\phi(e'_2e'_3))$ contains a subsequence of the form $(\ell_1, 1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$, for otherwise $S(\phi(\partial D_1'^+)) = S(\phi(e'_2e'_3)) = (1, 1)$ which contains neither S_1 nor (ℓ, S_2, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$, contradicting Lemma 3.21. It follows that a term 1 occurs in $CS(\phi(\partial D_1')) = CS(r)$, a contradiction.

(6) Let $S(z_{1,e}z'_{1,e}{}^{-1}) = (m, m+1)$ and $S(y'_{2,b}{}^{-1}y_{2,b}) = (m+1)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m, m)$ (see Figure 22(b)). Then $S(\phi(e'_2e'_3))$ contains a term $m+2$. Here, if $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ contains a term $m+2$, a contradiction to Proposition 7.2. On the other hand, if $J \subsetneq M$, then a term of the form $m+2+c$ with $c \in \mathbb{Z}_+ \cup \{0\}$ occurs in $CS(\phi(\partial D_1')) = CS(r)$, a contradiction.

(7) Let $S(z_{1,e}z'_{1,e}{}^{-1}) = (m+1, m+1)$ and $S(y'_{2,b}{}^{-1}y_{2,b}) = (m+1)$. Suppose on the contrary that $S(z_{1,e}y_{2,b}) = (m+1, m)$. Then $S(\phi(e'_2e'_3))$ contains a term $m+2$. So, arguing as in the proof of (6), we reach a contradiction. \square

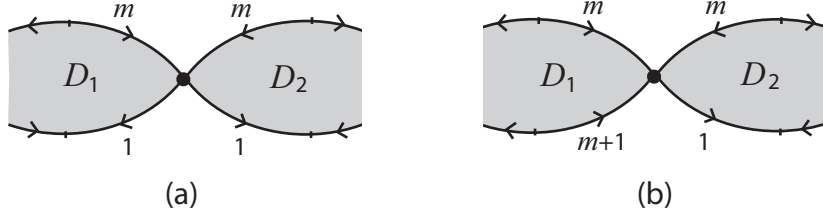


FIGURE 22. (a) Lemma 8.4(5), and (b) Lemma 8.4(6)

We are now ready to prove the following.

Proposition 8.5. *Assume that r is general. Under Hypothesis B or Hypothesis C, we may assume that the following hold.*

- (1) *For every face D_i of J , the following hold.*
 - (a) *If $S(\phi(\partial D_i^+))$ contains S_1 , then it contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.*
 - (b) *If $S(\phi(\partial D_i^-))$ contains S_1 , then it contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.*
- (2) *Every vertex x in J with $\deg_J(x) = 4$ is either converging or diverging.*

Proof. By Lemma 8.3, we may assume that (1) is satisfied. We prove that we can modify the annular diagram M so that it satisfies (2). Then the resulting annular diagram satisfies both (1) and (2), because Lemma 8.2 guarantees that if M satisfies (2) then it also satisfies (1).

Since (1) is satisfied, the subwords y_i and z_i of $\phi(\partial D_i^+)$ and the subwords y'_i and z'_i of $\phi(\partial D_i^-)$ are nonempty. Suppose on the contrary that there is a vertex $x \in J$ with $\deg_J(x) = 4$ such that x is neither converging nor diverging. We may assume x is the vertex between D_1 and D_2 . Then x has one of the five types as depicted in Figure 23, where c_i and d_i ($i = 1, 2$) are positive integers, up to simultaneous reversal of the edge orientations and up to the reflection in the vertical edge passing through the vertex x . To see this, let L be the set of labels of incoming unit segments of x , and orient each of the unit segment so that the associated label is equal to a or b as in Figure 15. If $L = \{a^{\pm 1}, b^{\pm 1}\}$, then we obtain the situation (a) or (b) in Figure 23. If L consists of three elements, then we may assume that a and a^{-1} , respectively, appear as the label of the upper left and lower right incoming unit segments and that b or b^{-1} does not belong to L . Then we obtain the situation (c) or (d) in Figure 23. If L consists of two elements, then we may assume both the upper left and lower right incoming unit segments have label a , and both

the upper left and lower right incoming unit segments have label b^{-1} , because x is not converging nor diverging. In this case, we have the situation (e) in Figure 23.

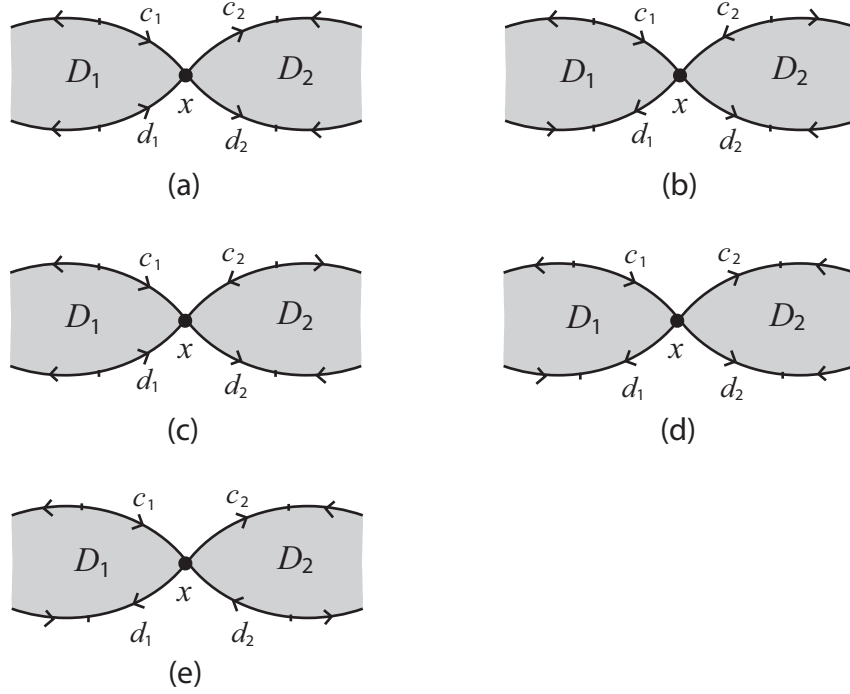


FIGURE 23. The five possible types of a vertex $x \in J$ with $\deg_J(x) = 4$ such that x is neither converging nor diverging.

Assume that x is as depicted in Figure 23(a). Then, for each $i = 1, 2$, c_i is a component of $CS(\phi(\partial D_i)) = CS(r)$ and so is equal to m or $m + 1$. Hence the component, $c_1 + c_2$, of $CS(\phi(\alpha)) = CS(s)$ is at least $2m$. This is a contradiction to Proposition 7.2.

Assume that x is as depicted in Figure 23(b). Then (c_1, c_2) is a subsequence of $CS(\phi(\alpha)) = CS(s)$. Since $CS(\phi(\alpha)) = CS(s)$ consists of m and $m + 1$ by Proposition 7.2, the only possibility is that $c_1 = c_2 = m$ and $d_1 = d_2 = 1$. So, we must have $S(z_{1,e}z'_{1,e}{}^{-1}) = S(y'_{2,b}{}^{-1}y_{2,b}) = (m + 1)$ and $S(z_{1,e}y_{2,b}) = (m, m)$. But this is impossible by Lemma 8.4(5).

Assume that x is as depicted in Figure 23(c). Then (c_1, c_2) is a subsequence of $CS(\phi(\alpha)) = CS(s)$ and hence each of c_1 and c_2 is either m or $m + 1$.

Moreover, $c_2 + d_2$ is a component of $CS(r)$ and hence it is either m or $m + 1$. So, we have the following four possibilities:

- (i) $c_1 = m, c_2 = m, d_1 = m, d_2 = 1$;
- (ii) $c_1 = m, c_2 = m, d_1 = m + 1, d_2 = 1$;
- (iii) $c_1 = m + 1, c_2 = m, d_1 = m, d_2 = 1$;
- (iv) $c_1 = m + 1, c_2 = m, d_1 = m + 1, d_2 = 1$.

However, (ii) and (iv) are impossible by Lemma 8.4(6) and Lemma 8.4(7), respectively. If (i) or (iii) happens, then $c_2 = d_1$. So we can transform M so that x is diverging as in Figure 24 (cf. the argument in the proof of Lemma 8.3 appealing to Figure 18).

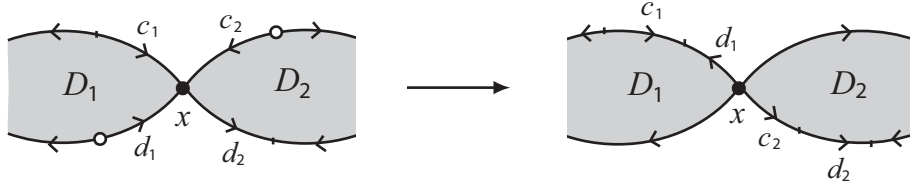


FIGURE 24. The transformation of Figure 23(c) when $c_2 = d_1$ so that x is diverging.

Assume that x is as depicted in Figure 23(d). Then $c_1 + c_2$ is a component of $CS(\phi(\alpha)) = CS(s)$ and hence it is either m or $m + 1$. Moreover, $c_1 + d_1$ is a component of $CS(r)$ and hence it is either m or $m + 1$. So, we have the following four possibilities:

- (i) $c_1 = 1, c_2 = m, d_1 = m - 1, d_2 = m$;
- (ii) $c_1 = 1, c_2 = m, d_1 = m - 1, d_2 = m + 1$;
- (iii) $c_1 = 1, c_2 = m, d_1 = m, d_2 = m$;
- (iv) $c_1 = 1, c_2 = m, d_1 = m, d_2 = m + 1$.

However, (i) and (ii) are impossible by Lemma 8.4(3) and Lemma 8.4(4), respectively. If (iii) or (iv) happens, then $c_2 = d_1$. So we can transform M so that x is converging as in Figure 25.

Assume that x is as depicted in Figure 23(e). Then $c_1 + c_2$ is a component of $CS(\phi(\alpha)) = CS(s)$ and hence it is either m or $m + 1$. Moreover, for each $i = 1, 2$, $c_i + d_i$ is a component of $CS(r)$ and hence it is either m or $m + 1$. So, we have the following eight possibilities:

- (i) $c_1 + c_2 = m, c_1 + d_1 = m, c_2 + d_2 = m$;
- (ii) $c_1 + c_2 = m, c_1 + d_1 = m, c_2 + d_2 = m + 1$;

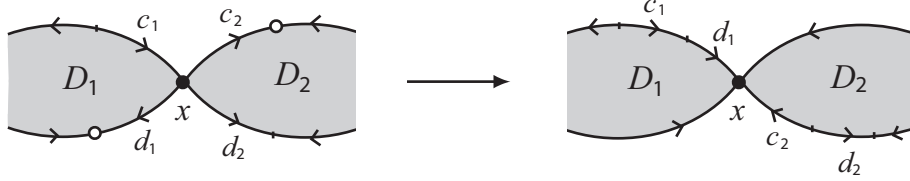


FIGURE 25. The transformation of Figure 23(d) when $c_2 = d_1$ so that x is converging.

- (iii) $c_1 + c_2 = m$, $c_1 + d_1 = m + 1$, $c_2 + d_2 = m$;
- (iv) $c_1 + c_2 = m$, $c_1 + d_1 = m + 1$, $c_2 + d_2 = m + 1$;
- (v) $c_1 + c_2 = m + 1$, $c_1 + d_1 = m$, $c_2 + d_2 = m$;
- (vi) $c_1 + c_2 = m + 1$, $c_1 + d_1 = m$, $c_2 + d_2 = m + 1$;
- (vii) $c_1 + c_2 = m + 1$, $c_1 + d_1 = m + 1$, $c_2 + d_2 = m$;
- (viii) $c_1 + c_2 = m + 1$, $c_1 + d_1 = m + 1$, $c_2 + d_2 = m + 1$.

However, (iv) and (v) are impossible by Lemma 8.4(2) and Lemma 8.4(1), respectively. If (i), (ii), (vii) or (viii) happens, then $c_2 = d_1$. Thus, as illustrated in Figure 25, we may transform M so that x is converging (cf. the argument in the proof of Lemma 8.3 appealing to Figure 18). If (iii) or (vi) happens, then $c_1 = d_2$. So we can transform M so that x is diverging as in Figure 24. \square

8.2. The case when vertices lie in an arbitrary layer. We next treat a general vertex x in M with $\deg_M(x) = 4$ by using induction on the number of layers of M . Note that the base step (i.e., the case when $M = J$) was already proved in Proposition 8.5(2). We need several lemmas and new notations.

Notation 8.6. Under Hypothesis A, suppose that the number of layers of M is $p + 1$ with $p \geq 0$. (Recall the characterization of M in Theorem 3.19.) For each $j = 0, 1, \dots, p$, we define J_j as follows: $J_0 = J$, and J_j is the outer boundary layer of $M - (J_0 \cup \dots \cup J_{j-1})$ for $j \geq 1$. Then $M = J_0 \cup J_1 \cup \dots \cup J_p$. We also define α_j and δ_j to be, respectively, outer and inner boundary cycles of J_j starting from a vertex lying in both the outer and inner boundaries of J_j . Let $\alpha_j = e_{j,1}, e_{j,2}, \dots, e_{j,2t}$ and $\delta_j^{-1} = e'_{j,1}, e'_{j,2}, \dots, e'_{j,2t}$ be the decompositions into oriented edges in ∂J_j . Then clearly for each $i = 1, \dots, t$, there is a face $D_{j,i}$ of J_j such that $e_{j,2i-1}, e_{j,2i}, e_{j,2i}^{-1}, e_{j,2i-1}^{-1}$ are consecutive edges in a boundary cycle of $D_{j,i}$. We denote the path $e_{j,2i-1}, e_{j,2i}$ by $\partial D_{j,i}^+$ and the path $e'_{j,2i-1}, e'_{j,2i}$ by $\partial D_{j,i}^-$. In particular, if $J_0 \cup \dots \cup J_j \subsetneq M$, then we may assume that $e'_{j,2i}$ and $e'_{j,2i+1}$ are two consecutive edges in $\partial D_{j+1,i} \cap \delta_j^{-1}$.

We can easily see that Lemmas 3.20 and 3.21 ([8, Lemmas 4.1 and 4.2]) hold not only for the faces of $J_0 = J$ but also for every face of M .

Lemma 8.7. *Under Hypothesis A and Notation 8.6, both of the following hold for every face $D_{j,i}$ of M .*

- (1) *None of $S(\phi(e_{j,2i-1}))$, $S(\phi(e_{j,2i}))$, $S(\phi(e'_{j,2i}))$ and $S(\phi(e'_{j,2i-1}))$ contains S_1 as a subsequence.*
- (2) *None of $S(\phi(e_{j,2i-1}))$, $S(\phi(e_{j,2i}))$, $S(\phi(e'_{j,2i}))$ and $S(\phi(e'_{j,2i-1}))$ contains a subsequence of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$.*

Lemma 8.8. *Under Hypothesis A and Notation 8.6, only one of the following holds for each face $D_{j,i}$ of M .*

- (1) *Both $S(\phi(\partial D_{j,i}^+))$ and $S(\phi(\partial D_{j,i}^-))$ contain S_1 as its subsequence.*
- (2) *Both $S(\phi(\partial D_{j,i}^+))$ and $S(\phi(\partial D_{j,i}^-))$ contain a subsequence of the form (ℓ, S_2, ℓ') , where $\ell, \ell' \in \mathbb{Z}_+$.*

In fact, these lemmas are proved by using Corollary 3.17 and the following facts:

- (i) The words $\phi(e_{j,i})$ and $\phi(e'_{j,i})$ are pieces. (If $1 \leq j \leq p-1$, this follows from the assumption that M is a reduced annular diagram. If $j = 0$ or p , this follows from [7, Convention 4.3].)
- (ii) The words $\phi(\partial D_{j,i}^+)$ and $\phi(\partial D_{j,i}^-)$ are not pieces. (Otherwise, the cyclic word $(\phi(\partial D_{j,i})) = (u_r^{\pm 1})$ becomes a product of three pieces, a contradiction to Proposition 3.14.)

Notation 8.9. Under Hypothesis A and Notation 8.6, Lemma 8.8 implies that we may decompose the word $\phi(\alpha_j)$ into

$$\phi(\alpha_j) \equiv y_{j,1}w_{j,1}z_{j,1}y_{j,2}w_{j,2}z_{j,2} \cdots y_{j,t}w_{j,t}z_{j,t},$$

where $\phi(\partial D_{j,i}^+) \equiv \phi(e_{j,2i-1}e_{j,2i}) \equiv y_{j,i}w_{j,i}z_{j,i}$, and where either $S(w_{j,i}) = S_1$ or both $S(w_{j,i}) = S_2$ and $y_{j,i}, z_{j,i}$ are nonempty words. We also have the decomposition of the word $\phi(\delta_j^{-1})$ as follows:

$$\phi(\delta_j^{-1}) \equiv y'_{j,1}w'_{j,1}z'_{j,1}y'_{j,2}w'_{j,2}z'_{j,2} \cdots y'_{j,t}w'_{j,t}z'_{j,t},$$

where $\phi(\partial D_{j,i}^-) \equiv \phi(e'_{j,2i-1}e'_{j,2i}) \equiv y'_{j,i}w'_{j,i}z'_{j,i}$, and where either $S(w'_{j,i}) = S_1$ or both $S(w'_{j,i}) = S_2$ and $y'_{j,i}, z'_{j,i}$ are nonempty words. Here, the indices for the 2-cells are considered modulo t , and the indices for the edges are considered modulo $2t$.

Lemma 8.10. *Assume that r is general. Under Hypothesis A and Notation 8.6, suppose that the vertex between $D_{j,i}$ and $D_{j,i+1}$ is either converging or diverging. Then none of the following occurs.*

- (1) $S(\phi(\partial D_{j,i}^+))$ ends with S_1 .
- (2) $S(\phi(\partial D_{j,i+1}^+))$ begins with S_1 .
- (3) $S(\phi(\partial D_{j,i}^-))$ ends with S_1 .
- (4) $S(\phi(\partial D_{j,i+1}^-))$ begins with S_1 .

Proof. We may assume that $i = 1$ and that the vertex is diverging. If $j = 0$, then the assertion is nothing other than Lemma 8.2. If $j = p$, then the assertion is proved by applying the proof of Lemma 8.2 to the inner boundary layer of M . So, we may assume $1 \leq j \leq p - 1$. Suppose on the contrary that (1) occurs, namely suppose that $S(\phi(\partial D_{j,1}^+))$ ends with S_1 . Then $S(\phi(\partial D_{j,1}^-))$ ends with (S_1, S_2) . Since the vertex is diverging, $S(\phi(\partial D_{j,1}^-)\phi(\partial D_{j,2}^-))$ contains a subsequence (S_1, S_2, d) for some $d \in \mathbb{Z}_+$. Thus we obtain a situation as illustrated in Figure 17(b), where $D_{j,1}$, $D_{j,2}$ and $D_{j+1,1}$, respectively, correspond to D_1 , D_2 and D'_1 in the figure. Then by the argument in the proof of Lemma 8.2 for the case where (1) occurs and $J \subsetneq M$, we see that the two 2-cells $D_{j,1}$ and $D_{j+1,1}$ form a reducible pair, a contradiction. (Here we use Lemma 8.7(1) instead of Lemma 3.20(1).) Hence (1) cannot occur. By a similar argument, we can see that (2), (3) and (4) cannot occur. \square

Now we are ready to prove the following generalization of Proposition 8.5.

Proposition 8.11. *Assume that r is general. Under Hypothesis A and Notation 8.6, we may assume that the following hold for every j .*

- (1) *For every face $D_{j,i}$ of J_j , the following hold.*
 - (a) *If $S(\phi(\partial D_{j,i}^+))$ contains S_1 , then it contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.*
 - (b) *If $S(\phi(\partial D_{j,i}^-))$ contains S_1 , then it contains a subsequence of the form (ℓ, S_1, ℓ') with $\ell, \ell' \in \mathbb{Z}_+$.*
- (2) *Every vertex x in J_j with $\deg_{J_j}(x) = 4$ is either converging or diverging.*

Proof. We simultaneously prove (1) and (2) by induction on $j \geq 0$. The base step $j = 0$ is already proved in Proposition 8.5. So fix $j \geq 1$. By the inductive hypothesis, $CS(\phi(\delta_{j-1}^{-1})) = CS(\phi(\alpha_j))$ consists of m and $m + 1$.

(1a) Suppose on the contrary that $S(\phi(\partial D_{j,1}^+))$ ends with S_1 so that $|z_{j,1}| = 0$ (see Notation 8.9). (The other case is treated similarly.) Then $S(z'_{j,1}) = S_2$ and so $S(z'_{j,1,e}) = (m)$. Note also that the assumption implies that the vertex between $D_{j,1}$ and $D_{j,2}$ is not converging nor diverging by Lemma 8.10. Thus,

arguing as in the proof of Lemma 8.3 when Lemma 8.2(1) does not hold, we see that $S(z'_{j,1,e}y'_{j,2}w'_{j,2})$ begins with a subsequence of the form $(m+d)$ with $d \in \mathbb{Z}_+$. Since $S(\phi(\partial D_{j,1}^+)) = CS(\phi(\delta_{j-1}^{-1}))$ ends with a term $m+1$, and since $CS(\phi(\alpha_j))$ consists of m and $m+1$ by the inductive hypothesis, the only possibility is that $d=1$ and $S(w_{j,1,e}y_{j,2,b}) = (m+1, m)$. Then, as illustrated in Figure 18, we may transform M so that $S(\phi(\partial D_{j,1}^+))$ ends with (S_1, m) . Since the new vertex of M is either converging or diverging, we see by Lemma 8.10 that none of the four (prohibited) conditions in the lemma holds. Thus by repeating this argument at every degree 4 vertex of J_j , we obtain the desired result.

(1b) Suppose on the contrary that $S(\phi(\partial D_{j,1}^-))$ ends with S_1 . (The other case is treated similarly.) Then $|z'_{j,1}| = 0$. It follows that $S(z_{j,1}) = S_2$, so that $S(z_{j,1,e}) = (m)$. Hence the sequence $S(z_{j,1,e}y_{j,2}w_{j,2})$ begins with a subsequence of the form either (m, d) or $(m+d)$, where $d \in \mathbb{Z}_+$. Here, if $S(z_{j,1,e}y_{j,2}w_{j,2})$ begins with a subsequence of the form (m, d) , then we see by an argument as in the proof of Lemma 8.2(1) that two 2-cells $D_{j,1}$ and $D_{j-1,2}$ are a reducible pair, contradicting that M is a reduced diagram. So $S(z_{j,1,e}y_{j,2}w_{j,2})$ must begin with a subsequence of the form $(m+d)$. Since $S(\phi(\partial D_{j,1}^+))$ ends with a term m , and since $S(\phi(\partial D_{j,1}^+)) = CS(\phi(\delta_{j-1}^{-1}))$ consists of m and $m+1$ by the inductive hypothesis, the only possibility is that $d=1$ and $S(w'_{j,1,e}y'_{j,2,b}) = (m+1, m)$. Then, as illustrated in Figure 19, we may transform M so that $S(\phi(\partial D_{j,1}^-))$ ends with (S_1, m) . Since the new vertex of M is converging or diverging, we see by Lemma 8.10 that none of the four (prohibited) conditions in the lemma holds. Thus by repeating this argument at every degree 4 vertex of J_j , we obtain the desired result.

(2) As in the proof of Proposition 8.5, we show that we can modify M so that it satisfies (2). Then it continues to satisfy (1) by Lemma 8.10. To this end, note that $CS(\phi(\alpha_j))$ consists of m and $m+1$ by the inductive hypothesis. By using this fact, we can see that the statement of Lemma 8.4 holds, where $z_{i,e}$, $z'_{i,e}$, $y_{i+1,e}$ and $y'_{i+1,e}$ are replaced with $z_{j,i,e}$, $z'_{j,i,e}$, $y_{j,i+1,e}$ and $y'_{j,i+1,e}$, respectively. (The proof of Lemma 8.4 works if we appeal to the fact that $CS(\phi(\alpha_j))$, instead of $CS(\phi(\alpha))$, consists of m and $m+1$.) So we can follow the proof of Proposition 8.5 and show that (2) holds. \square

Corollary 8.12. *Assume that r is general. Under Hypothesis A and Notation 8.6, we may assume that the following hold.*

- (1) *Every vertex x in M with $\deg_M(x) = 4$ is either converging or diverging.*
- (2) *For every face $D_{j,i}$ of M , one of the following hold.*

- (a) Both $S(\phi(\partial D_{j,i}^+))$ and $S(\phi(\partial D_{j,i}^-))$ contain (m, S_1, m) as a subsequence.
- (b) Both $S(\phi(\partial D_{j,i}^+))$ and $S(\phi(\partial D_{j,i}^-))$ contain $(m+1, S_2, m+1)$ as a subsequence.
- (3) For every j , both $CS(\phi(\alpha_j))$ and $CS(\phi(\delta_j^{-1}))$ consist of m and $m+1$.

Proof. (1) is nothing other than Proposition 8.11(2). (2) follows from Lemma 8.8, Proposition 8.11(1) and the fact that S_1 (resp. S_2) begins and ends with $m+1$ (resp. m). \square

9. KEY RESULTS FOR THE INDUCTION

In this section, we prove key results, Propositions 9.3 and 9.8 for $r = [m, m_2, \dots, m_k]$ with $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$ and $r = [m, 1, m_3, \dots, m_k]$ with $m \geq 2$ and $k \geq 4$, respectively, used for the inductive proof of Main Theorem 2.5 in Section 10. Throughout this section, we assume that Hypothesis A holds and that the annular diagram M satisfies the conditions in Corollary 8.12.

9.1. The case for $r = [m, m_2, \dots, m_k]$ with $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. We first establish a key result, Proposition 9.3, for $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Recall from Remark 7.1(1) that $CS(r) = ((S_1, S_2, S_1, S_2))$, where S_1 begins and ends with $(m+1, (m_2-1)\langle m \rangle, m+1)$, and S_2 begins and ends with $(m_2\langle m \rangle)$.

Lemma 9.1. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Under Hypothesis A and Notation 8.9, the following hold for every i and j .*

- (1) $S(z_{j,i,e}y_{j,i+1,b}) \neq (m+1, m+1)$.
- (2) $S(z'_{j,i,e}y'_{j,i+1,b}) \neq (m+1, m+1)$.

Proof. We prove only (1), because the proof of (2) is parallel. Suppose on the contrary that $S(z_{j,1,e}y_{j,2,b}) = (m+1, m+1)$ for some j . First assume $j = 0$. If Hypothesis B holds, then $S(z_{0,1})$ begins with $(m_2\langle m \rangle)$, because S_2 begins and ends with $(m_2\langle m \rangle)$ (see Remark 7.1(1)) whereas $S(z_{0,1,e}) = (m+1)$ by assumption. This implies that $CS(\phi(\alpha_0)) = CS(s)$ contains two consecutive m 's. So $CS(s)$ contains two consecutive m 's and two consecutive $m+1$'s, contradicting Lemma 3.5. On the other hand, if Hypothesis C holds, then $CS(s)$ contains two consecutive m 's (because it contains S_2) and two consecutive $m+1$'s by assumption, again contradicting Lemma 3.5. Next assume $j \geq 1$. By using Lemma 8.7, we can see that $S(\phi(e_{j,2}e_{j,3}))$ contains

a subsequence $(m+1, m+1)$. Thus $CS(\phi(\partial D_{j-1,2})) = CS(r)$ contains two consecutive $m+1$'s, a contradiction. \square

Corollary 9.2. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Under Hypothesis A and Notation 8.6, the following hold for every j .*

- (1) $CS(\phi(\alpha_j))$ does not contain $(m+1, m+1)$ as a subsequence.
- (2) $CS(\phi(\delta_j^{-1}))$ does not contain $(m+1, m+1)$ as a subsequence.

Proof. We prove only (1), because the proof of (2) is parallel. Suppose on the contrary that $CS(\phi(\alpha_j))$ contains a subsequence $(m+1, m+1)$ for some j . Let v be a subword of the cyclic word $(\phi(\alpha_j))$ corresponding to a subsequence $(m+1, m+1)$. Note that $S(\phi(\partial D_{j,i}^+))$ does not contain $(m+1, m+1)$, because $S(r) = S(\phi(\partial D_{j,i}))$ does not. Thus the only possibility is that $v = z_{j,i,e} y_{j,i+1,b}$ for some i by Corollary 8.12. But this is impossible by Lemma 9.1(1). \square

Proposition 9.3. *Let $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Suppose that there are two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$ such that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely suppose that Hypothesis A holds. Let $\tilde{r} = [m_2-1, m_3, \dots, m_k]$ be as in Lemma 3.7. Then there are two distinct rational numbers $\tilde{s}, \tilde{s}' \in I_1(\tilde{r}) \cup I_2(\tilde{r})$ such that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ are homotopic in $S^3 - K(\tilde{r})$. Moreover, there is a reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ such that none of the degree 4 vertices is mixing.*

Proof. Recall from Corollaries 8.12 and 9.2 that both $CS(\phi(\alpha_j))$ and $CS(\phi(\delta_j^{-1}))$ consist of m and $m+1$ without a subsequence $(m+1, m+1)$ for every j . In particular, both $CS(\phi(\alpha_0)) = CS(s)$ and $CS(\phi(\delta_p^{-1})) = CS(s')$ consist of m and $m+1$ without a subsequence $(m+1, m+1)$. This implies that if $s = [p_1, p_2, \dots, p_h]$ and $s' = [q_1, q_2, \dots, q_l]$, where $p_i, q_j \in \mathbb{Z}_+$ and $p_h, q_l \geq 2$, then $p_1 = q_1 = m$ and $p_2, q_2 \geq 2$. Put $\tilde{s} = [p_2-1, p_3, \dots, p_h]$ and $\tilde{s}' = [q_2-1, q_3, \dots, q_l]$ as in Lemma 3.7.

Claim. *Both \tilde{s} and \tilde{s}' belong to $I_1(\tilde{r}) \cup I_2(\tilde{r})$.*

Proof of Claim. Since $p_1 = q_1 = m$, we have

$$\tilde{s} = \frac{1}{\frac{1}{-p_1 + \frac{1}{s}} - 1} = \frac{1}{\frac{1}{-m + \frac{1}{s}} - 1} \quad \text{and} \quad \tilde{s}' = \frac{1}{\frac{1}{-q_1 + \frac{1}{s'}} - 1} = \frac{1}{\frac{1}{-m + \frac{1}{s'}} - 1}.$$

Put $I_1(r) = [0, r_1]$ and $I_2(r) = [r_2, 1]$. Recall from Section 2 that

$$r_1 = \begin{cases} [m, m_2, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m, m_2, \dots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

Also put $I_1(\tilde{r}) = [0, t_1]$ and $I_2(\tilde{r}) = [t_2, 1]$; then we have

$$t_1 = \begin{cases} [m_2 - 1, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_2 - 1, \dots, m_{k-1}] & \text{if } k \text{ is even,} \end{cases}$$

$$t_2 = \begin{cases} [m_2 - 1, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_2 - 1, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even.} \end{cases}$$

It then follows that

$$t_1 = \frac{1}{\frac{1}{-m + \frac{1}{r_2}} - 1} \quad \text{and} \quad t_2 = \frac{1}{\frac{1}{-m + \frac{1}{r_1}} - 1}.$$

Therefore the fact $s, s' \in I_1(r) \cup I_2(r)$ yields $\tilde{s}, \tilde{s}' \in I_1(\tilde{r}) \cup I_2(\tilde{r})$. \square

Let \tilde{R} be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{\tilde{r}}$ of the upper presentation $G(K(\tilde{r})) = \langle a, b \mid u_{\tilde{r}} \rangle$. Then as described below, we can construct a reduced annular \tilde{R} -diagram \tilde{M} such that $u_{\tilde{s}}$ is an outer boundary label and $u_{\tilde{s}'}^{\pm 1}$ is an inner boundary label of \tilde{M} . This proves that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ are homotopic in $S^3 - K(\tilde{r})$. \square

We shall describe the explicit construction of a reduced annular \tilde{R} -diagram \tilde{M} from M . To this end, we introduce the following definition.

Definition 9.4. Suppose $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$. Let w be an alternating word in $\{a, b\}$, and suppose that $S(w) = (a_1, a_2, \dots, a_k)$ is a finite sequence consisting of m and $m+1$, which does not contain $(m+1, m+1)$. Then we define the T -sequence of w , denoted by $T(w)$, and the cyclic T -sequence, denoted by $CT(w)$, as follows. Express $S(w)$ as

$$(*, t_1 \langle m \rangle, m+1, t_2 \langle m \rangle, \dots, m+1, t_s \langle m \rangle, *'),$$

where each of $*$ and $*$ ' is either $m+1$ or \emptyset and (t_1, t_2, \dots, t_s) is a sequence of positive integers. Then $T(w)$ is defined to be the sequence (t_1, \dots, t_s) . If

precisely one of $*$ and $*$ ' is $m+1$ and the other is \emptyset , we define $CT(w)$ to be the cyclic sequence $((t_1, \dots, t_s))$. If this w represents a reduced cyclic word $u = (w)$, then we define the cyclic sequence $CT(u)$ by $CT(w)$.

Under Hypothesis A and Notation 8.6, by Corollaries 8.12 and 9.2, both $CS(\phi(\alpha_j))$ and $CS(\phi(\delta_j^{-1}))$ consist of m and $m+1$ without a subsequence $(m+1, m+1)$, so the cyclic sequences $CT(\phi(\alpha_j))$ and $CT(\phi(\delta_j^{-1}))$ are well-defined for every j . Recall that every vertex in M of degree 4 is assumed to be converging or diverging by Corollary 8.12. Moreover, we can also assume, by using Corollary 8.12(2), that every degree 2 vertex of M is also either converging or diverging. (In the new diagram, it may happen that some $\phi(e_{0,i})$ or some $\phi(e'_{p,i})$ is not a piece and so [7, Convention 4.6] is not satisfied. But this does not affect the arguments in this section.) Hence $S(\phi(e_{j,i}))$ is a subsequence of $S(\phi(\alpha_j))$, which consists of m and $m+1$ and does not contain $(m+1, m+1)$ as a subsequence. Thus the T -sequence of $\phi(e_{j,i})$ is also well-defined for every i and j . Similarly the T -sequence of $\phi(e'_{j,i})$ is also well-defined for every i and j . By Corollary 8.12(2), we may assume that $T(\phi(e_{j,i}))$ and $T(\phi(e'_{j,i}))$ are nonempty for every i and j .

Now we construct a reduced annular \tilde{R} -diagram (\tilde{M}, ψ) from the annular R -diagram (M, ϕ) by taking T -sequences of the boundary labels, as follows. Take the underlying map of \tilde{M} being the same as that of M . For every i and j , by $\tilde{e}_{j,i}$ denote the edge of \tilde{M} which corresponds to the edge $e_{j,i}$ of M , and assign an alternating word, $\psi(\tilde{e}_{j,i})$, in $\{a, b\}$ to $\tilde{e}_{j,i}$ in the following order.

Step 1. For each $i = 1, \dots, 2t$, assign $\psi(\tilde{e}_{0,i})$ so that $\psi(\tilde{e}_{0,1} \cdots \tilde{e}_{0,i}) := \psi(\tilde{e}_{0,1}) \cdots \psi(\tilde{e}_{0,i})$ is an alternating word such that

$$S(\psi(\tilde{e}_{0,1} \cdots \tilde{e}_{0,i})) = T(\phi(e_{0,1} \cdots e_{0,i})).$$

Once this assignment is done, we see the following.

- (i) The word $\psi(\tilde{e}_{0,1} \cdots \tilde{e}_{0,2t})$ is cyclically alternating, because the sum of the components of $CT(s) = CS(\tilde{s})$ is even.
- (ii) $CS(\psi(\tilde{e}_{0,1} \cdots \tilde{e}_{0,2t})) = CT(\phi(e_{0,1} \cdots e_{0,2t})) = CT(\phi(\alpha_0)) = CT(s) = CS(\tilde{s})$, because $CS(\tilde{s})$ has even number of components.

Step 2. Assign $\psi(\tilde{e}'_{0,1})$ so that $\psi(\tilde{e}_{0,1}^{-1} \tilde{e}'_{0,1})$ is an alternating word such that $S(\psi(\tilde{e}_{0,1}^{-1} \tilde{e}'_{0,1})) = T(\phi(e_{0,1}^{-1} e'_{0,1}))$, and assign $\psi(\tilde{e}'_{0,2})$ so that $\psi(\tilde{e}'_{0,1} \tilde{e}'_{0,2})$ is an alternating word such that $S(\psi(\tilde{e}'_{0,1} \tilde{e}'_{0,2})) = T(\phi(e'_{0,1} e'_{0,2}))$. Once this assignment is done, we see the following.

- (i) The word $\psi(\tilde{e}_{0,1} \tilde{e}_{0,2} \tilde{e}'_{0,2}^{-1} \tilde{e}'_{0,1}^{-1})$ is cyclically alternating, because the sum of the components of $CT(r) = CS(\tilde{r})$ is even.

- (ii) $CS(\psi(\tilde{e}_{0,1}\tilde{e}_{0,2}\tilde{e}_{0,2}'^{-1}\tilde{e}_{0,1}'^{-1})) = CT(\psi(e_{0,1}e_{0,2}e_{0,2}'^{-1}e_{0,1}'^{-1})) = CT(r) = CS(\tilde{r})$,
because $CS(\tilde{r})$ has even number of components.

Step 3. For $i = 2, \dots, t$, assign $\psi(\tilde{e}_{0,2i-1})$ so that $\psi(\tilde{e}_{0,2i-1}^{-1}\tilde{e}_{0,2i-1}')$ is an alternating word such that $S(\psi(\tilde{e}_{0,2i-1}^{-1}\tilde{e}_{0,2i-1}')) = T(\phi(e_{0,2i-1}^{-1}e_{0,2i-1}'))$, and assign $\psi(\tilde{e}_{0,2i}')$ so that $\psi(\tilde{e}_{0,2i-1}'\tilde{e}_{0,2i}') is an alternating word such that $S(\psi(\tilde{e}_{0,2i-1}'\tilde{e}_{0,2i}')) = T(\phi(e_{0,2i-1}'e_{0,2i}'))$. Then we see the following.$

- (i) $S(\psi(\tilde{e}_{0,1}' \cdots \tilde{e}_{0,2i-1}')) = T(\phi(e_{0,1}' \cdots e_{0,2i-1}'))$, because of the reason described after this list.
- (ii) The word $\psi(\tilde{e}_{0,2i-1}\tilde{e}_{0,2i}\tilde{e}_{0,2i}'^{-1}\tilde{e}_{0,2i-1}'^{-1})$ is cyclically alternating, because the sum of the components of $CT(r) = CS(\tilde{r})$ is even.
- (iii) $CS(\psi(\tilde{e}_{0,2i-1}\tilde{e}_{0,2i}\tilde{e}_{0,2i}'^{-1}\tilde{e}_{0,2i-1}'^{-1})) = CT(\phi(e_{0,2i-1}e_{0,2i}e_{0,2i}'^{-1}e_{0,2i-1}'^{-1})) = CT(r) = CS(\tilde{r})$, because $CS(\tilde{r})$ has even number of components.
- (iv) The word $\psi(\tilde{e}_{0,1}' \cdots \tilde{e}_{0,2t}')$ is cyclically alternating, because the sum of the components of $CT(\phi(\delta_0))$ is even.
- (v) $CS(\psi(\tilde{e}_{0,1}' \cdots \tilde{e}_{0,2t}')) = CT(\phi(e_{0,1}' \cdots e_{0,2t}')) = CT(\phi(\delta_0))$, because $CT(\phi(\delta_0))$ has even number of components.

In the above, condition (i) is verified as follows. We explain the reason when $i = 2$. (The other cases can be treated similarly.) Since $CS(\phi(\alpha_0)) = CS(s)$ and $CS(\phi(\partial D_1)) = CS(\phi(\partial D_2)) = CS(r)$ consist of m and $m + 1$ and do not contain $(m + 1, m + 1)$, we have the four possibilities around the vertex between D_1 and D_2 as described in the left figures in Figure 26, up to reflection in the vertical line passing through the vertex. In each of the right figure, we may assume without loss of generality that the upper left segment is oriented so that it is converging into the vertex. Then the orientations of the three remaining segments in each of the right figures are specified by the requirements in Steps 1 and 2 and the new requirement $S(\psi(\tilde{e}_{0,3}^{-1}\tilde{e}_{0,3}')) = T(\phi(e_{0,3}^{-1}e_{0,3}'))$. In each case, we can check that the condition $S(\psi(\tilde{e}_{0,1}'\tilde{e}_{0,2}\tilde{e}_{0,3}')) = T(\phi(e_{0,1}'e_{0,2}e_{0,3}'))$ holds.

Step 4. For each $j = 1, \dots, p$, repeat Steps 2 and 3 to obtain the following.

- (i) The word $\psi(\tilde{e}_{j,2i-1}\tilde{e}_{j,2i}\tilde{e}_{j,2i}'^{-1}\tilde{e}_{j,2i-1}'^{-1})$ is cyclically alternating for every $i = 1, \dots, t$.
- (ii) $CS(\psi(\tilde{e}_{j,2i-1}\tilde{e}_{j,2i}\tilde{e}_{j,2i}'^{-1}\tilde{e}_{j,2i-1}'^{-1})) = CT(\phi(e_{j,2i-1}e_{j,2i}e_{j,2i}'^{-1}e_{j,2i-1}'^{-1})) = CT(r) = CS(\tilde{r})$ for every $i = 1, \dots, t$.
- (iii) The word $\psi(\tilde{e}_{p,1}' \cdots \tilde{e}_{p,2t}')$ is cyclically alternating.
- (iv) $CS(\psi(\tilde{e}_{p,1}' \cdots \tilde{e}_{p,2t}')) = CT(\phi(e_{p,1}' \cdots e_{p,2t}')) = CT(\psi(\delta_p)) = CT(s') = CS(\tilde{s}')$.

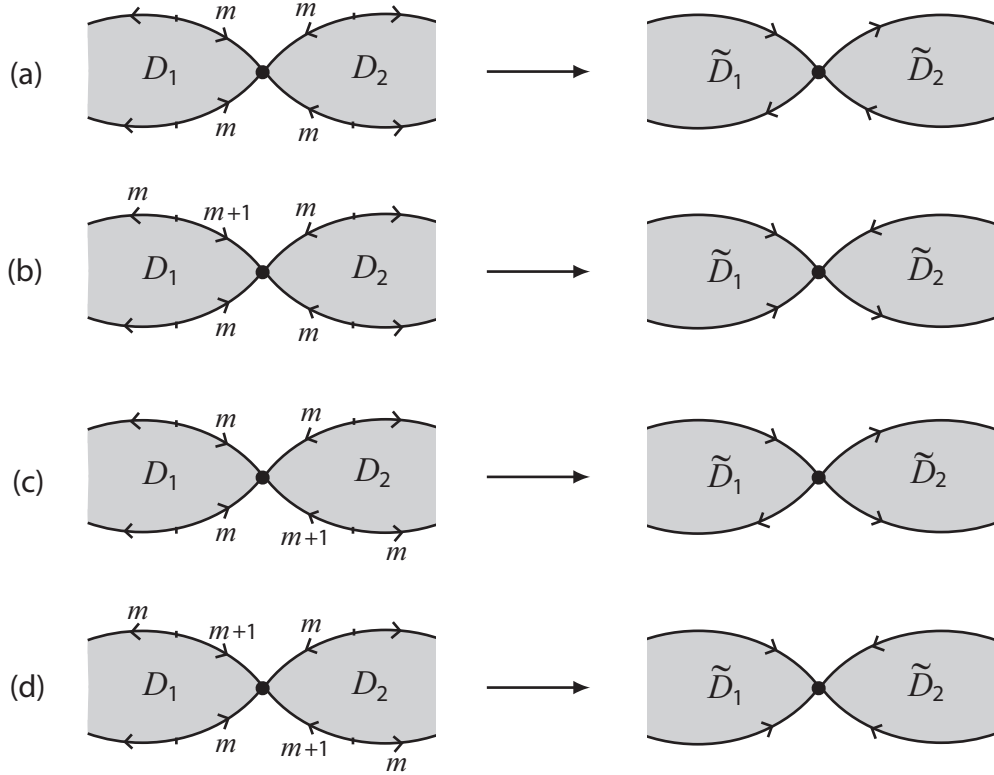


FIGURE 26. Step 3 of the construction of \tilde{M} from M

It is obvious from the construction (see Figure 26) that none of the degree 4 vertices of the diagram \tilde{M} constructed above is mixing.

Finally we show that \tilde{M} is reduced. Suppose on the contrary that \tilde{M} is not reduced. Then there is a pair of faces, say \tilde{D} and \tilde{D}' , in \tilde{M} having a common edge, say $\tilde{e} = \partial\tilde{D} \cap \partial\tilde{D}'$, such that $\psi(\delta_1) \equiv \psi(\delta_2)^{-1}$, where $\tilde{e}\delta_1$ and $\delta_2\tilde{e}^{-1}$ are boundary cycles of \tilde{D} and \tilde{D}' , respectively. Then we see that the corresponding faces D and D' in M have a common edge $e = \partial D \cap \partial D'$ such that $S(\phi(e\gamma_1)) = S(\phi(e\gamma_2^{-1}))$, where $e\gamma_1$ and γ_2e^{-1} are boundary cycles of D and D' , respectively. So two words $\phi(e\gamma_1) \equiv \phi(e)\phi(\gamma_1)$ and $\phi(e\gamma_2^{-1}) \equiv \phi(e)\phi(\gamma_2)^{-1}$ have the same initial letter and the same associated S -sequence. Then by Lemma 3.2 $\phi(e)\phi(\gamma_1) \equiv \phi(e)\phi(\gamma_2)^{-1}$, so $\phi(\gamma_1) \equiv \phi(\gamma_2)^{-1}$, which contradicts the fact that M is reduced.

9.2. **The case for** $r = [m, 1, m_3, \dots, m_k]$ **with** $m \geq 2$ **and** $k \geq 4$. We next establish a key result, Proposition 9.8, for $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Recall from Remark 7.1(2) that $CS(r) = ((S_1, S_2, S_1, S_2))$, where S_1 begins and ends with $((m_3 + 1)\langle m + 1 \rangle)$, and S_2 begins and ends with $(m, m_3\langle m + 1 \rangle, m)$.

Lemma 9.5. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Under Hypothesis A and Notation 8.9, the following hold for every i and j .*

- (1) $S(z_{j,i,e}y_{j,i+1,b}) \neq (m, m)$.
- (2) $S(z'_{j,i,e}y'_{j,i+1,b}) \neq (m, m)$.

Proof. We prove only (1), because the proof of (2) is parallel. Suppose on the contrary that $S(z_{j,1,e}y_{j,2,b}) = (m, m)$ for some j . First assume $j = 0$. If Hypothesis B holds, then $CS(\phi(\alpha_0)) = CS(s)$ contains two consecutive $m + 1$'s (because it contains S_1) and two consecutive m 's by assumption. This contradicts Lemma 3.5. On the other hand, if Hypothesis C holds, then $S(z_{0,1,e})$ begins with $((m_3 + 1)\langle m + 1 \rangle)$, since S_1 begins and ends with $((m_3 + 1)\langle m + 1 \rangle)$ whereas $S(z_{0,1,e}) = m$ by assumption. This implies that $CS(s)$ contains two consecutive m 's and two consecutive $m + 1$'s, again contradicting Lemma 3.5. Next assume $j \geq 1$. By using Lemma 8.7, we can see that $S(\phi(e_{j,2}e_{j,3}))$ contains a subsequence of the form (ℓ_1, m, m, ℓ_2) with $\ell_1, \ell_2 \in \mathbb{Z}_+$. Hence $CS(\phi(\partial D_{j-1,2})) = CS(r)$ contains two consecutive m 's, a contradiction. \square

Corollary 9.6. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Under Hypothesis A and Notation 8.6, the following hold for every j .*

- (1) *No two consecutive terms of $CS(\phi(\alpha_j))$ can be (m, m) .*
- (2) *No two consecutive terms of $CS(\phi(\delta_j^{-1}))$ can be (m, m) for every $j = 0, \dots, p$.*

Proof. We prove only (1), because the proof of (2) is parallel. Suppose on the contrary that $CS(\phi(\alpha_j))$ contains a subsequence (m, m) for some j . Let v be a subword of the cyclic word $(\phi(\alpha_j))$ corresponding to a subsequence (m, m) . Note that $S(\phi(\partial D_{j,i}^+))$ does not contain (m, m) , because $S(r) = S(\phi(\partial D_{j,i}))$ does not. Thus the only possibility is that $v = z_{j,i,e}y_{j,i+1,b}$ for some i by Corollary 8.12. But this is impossible by Lemma 9.5(1). \square

Definition 9.7. Suppose $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Let w be an alternating word in $\{a, b\}$, and suppose that $S(w) = (a_1, a_2, \dots, a_k)$ is a finite sequence consisting of m and $m + 1$, which does not contain (m, m) .

Then we define the T -sequence of w , denoted by $T(w)$, and the cyclic T -sequence, denoted by $CT(w)$, as follows. Express $S(w)$ as

$$(*, t_1 \langle m+1 \rangle, m, t_2 \langle m+1 \rangle, \dots, m, t_s \langle m+1 \rangle, *'),$$

where each of $*$ and $*$ ' is either m or \emptyset and (t_1, t_2, \dots, t_s) is a sequence of positive integers. Then $T(w)$ is defined to be the sequence (t_1, \dots, t_s) . If precisely one of $*$ and $*$ ' is m and the other is \emptyset , we define $CT(w)$ to be the cyclic sequence $((t_1, \dots, t_s))$. If this w represents a reduced cyclic word $u = (w)$, then we define the cyclic sequence $CT(u)$ by $CT(w)$.

Under Hypothesis A and Notation 8.6, by Corollaries 8.12 and 9.6, both $CS(\phi(\alpha_j))$ and $CS(\phi(\delta_j^{-1}))$ consist of m and $m+1$ without a subsequence (m, m) , so the cyclic sequences $CT(\phi(\alpha_j))$ and $CT(\phi(\delta_j^{-1}))$ are well-defined for every j . Clearly we may assume that every degree 2 vertex of M is either converging or diverging. Moreover since every vertex in M of degree 4 is assumed to be converging or diverging by Corollary 8.12, the T -sequence of $\phi(e_{j,i})$ is also well-defined for every i and j . Then as in the previous case, we can construct an annular \tilde{R} -diagram \tilde{M} from the annular R -diagram M by taking T -sequences of the boundary labels.

Proposition 9.8. *Let $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Suppose that there are two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$ such that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely suppose that Hypothesis A holds. Let $\tilde{r} = [m_3, \dots, m_k]$ be as in Lemma 3.7. Then there are two distinct rational numbers $\tilde{s}, \tilde{s}' \in I_1(\tilde{r}) \cup I_2(\tilde{r})$ such that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ are homotopic in $S^3 - K(\tilde{r})$. Moreover, there is a reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ such that none of the degree 4 vertices is mixing.*

Proof. Recall from Corollaries 8.12 and 9.6 that both $CS(\phi(\alpha_j))$ and $CS(\phi(\delta_j^{-1}))$ consist of m and $m+1$ without a subsequence (m, m) , for every j . In particular, both $CS(\phi(\alpha_0)) = CS(s)$ and $CS(\phi(\delta_p^{-1})) = CS(s')$ consist of m and $m+1$ without a subsequence (m, m) . This implies that if $s = [p_1, p_2, \dots, p_h]$ and $s' = [q_1, q_2, \dots, q_l]$, where $p_i, q_j \in \mathbb{Z}_+$ and $p_h, q_l \geq 2$, then $p_1 = q_1 = m$, $p_2 = q_2 = 1$ and $h, l \geq 3$. Put $\tilde{s} = [p_3, \dots, p_h]$ and $\tilde{s}' = [q_3, \dots, q_l]$ as in Lemma 3.7. Let \tilde{R} be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{\tilde{r}}$ of the upper presentation $G(K(\tilde{r})) = \langle a, b \mid u_{\tilde{r}} \rangle$. Then, as in the previous case, we can construct a reduced annular \tilde{R} -diagram (\tilde{M}, ψ) such that $u_{\tilde{s}}$ is an outer boundary label and $u_{\tilde{s}'}^{\pm 1}$ is an inner boundary label of \tilde{M} . This proves that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ are homotopic in $S^3 - K(\tilde{r})$.

Moreover, we can also see that \tilde{M} is reduced and that none of the degree 4 vertices of the diagram \tilde{M} constructed above is mixing. \square

10. PROOF OF MAIN THEOREM 2.5 FOR THE GENERAL CASES

In this section, we prove Main Theorem 2.5 when r is general. To this end, we use the following terminology.

- (1) A rational number r with $0 < r \leq 1/2$ is *special*, if it is equal to $1/p = [p]$ with $p \geq 2$, $[m, n]$, or $[m, 1, n]$ with $m, n \geq 2$.
- (2) A rational number r with $1/2 < r < 1$ is *special*, if $1 - r$ is special.
- (3) A rational number r with $0 < r < 1$ is *general*, if it is not special.

The following proposition forms the starting point of the inductive proof of Main Theorem 2.5.

Proposition 10.1. *Let r be a special rational number with $0 < r < 1$. Then the following is the complete list of pairs of mutually distinct elements $\{s, s'\}$ of $I_1(r) \cup I_2(r)$ such that α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

- (1) $r = 1/p$ and the set $\{s, s'\}$ equals $\{q_1/p_1, q_2/p_2\}$, where $p \geq 2$ is an integer, and $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.
- (2) $r = 3/8 = [2, 1, 2]$ and the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.
- (3) $r = 1 - 1/p = [1, p - 1]$ and the set $\{s, s'\}$ equals $\{1 - q_1/p_1, 1 - q_2/p_2\}$, where $p \geq 2$ is an integer, and $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.
- (4) $r = 1 - 3/8 = [1, 1, 1, 2]$ and the set $\{s, s'\}$ equals either $\{1 - 1/6, 1 - 3/10\}$ or $\{1 - 3/4, 1 - 5/12\}$.

Moreover, any reduced conjugacy diagram over $G(K(r))$ for α_s and $\alpha_{s'}$ contains a vertex which is mixing.

Proof. Suppose $0 < r \leq 1/2$. Then by Theorem 2.3, Corollary 2.4 and the results in Sections 5 and 6, we see that (1) or (2) holds. We prove the assertion for the conjugacy diagram in this case. Suppose that (1) holds. Then the outer boundary layer of the conjugacy diagram should be as depicted in Figure 27 by [7, Section 6]. Since all vertices of the diagram are mixing, we obtain the desired result. Suppose that (2) holds. Then, by [8, Section 7, in particular Figures 23(a) and 25(a)], the conjugacy diagram should be as depicted in Figure 28. Again, every vertex of degree 4 is mixing, and so we obtain the desired result.

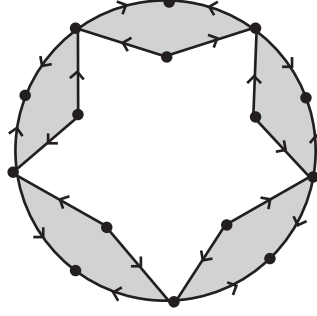


FIGURE 27. The outer boundary layer of any of the conjugacy diagrams for the case $r = 1/p$.

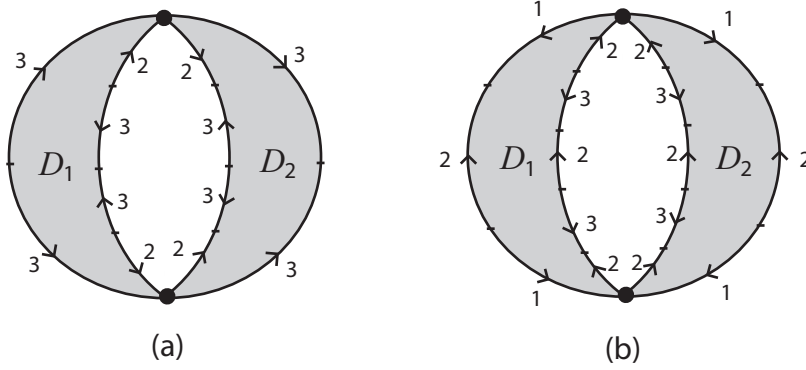


FIGURE 28. The conjugacy diagrams for the case $r = [2, 1, 2]$, where $\{s, s'\}$ is (a) $\{1/6, 3/10\}$ and (b) $\{3/4, 5/12\}$.

Suppose $1/2 < r < 1$. Note that there is a homeomorphism $f : (S^3, K(r)) \rightarrow (S^3, K(1-r))$ preserving the bridge sphere such that $f(\alpha_s) = f(\alpha_{1-s})$ and that f induces an isomorphism from $G(K(r))$ to $G(K(1-r))$ sending the standard generators a and b to a and b^{-1} , respectively. In fact, such a homeomorphism is obtained as the composition of the natural homeomorphisms

$$(S^3, K(r)) \rightarrow (S^3, K(-r)) \rightarrow (S^3, K(1-r)),$$

where the latter homeomorphism is explained in [6, the end of Section 3]. Thus, by Theorem 2.3 and the results in Sections 5 and 6, we see that (3) or (4) holds. Moreover, the conjugacy diagram over $G(K(r))$ is obtained as the isomorphic image of the conjugacy diagram over $G(K(1-r))$. Since the image

of a mixing vertex by the isomorphism is again a mixing vertex, we obtain the last assertion of the proposition. \square

Proof of Main Theorem 2.5 for the case when r is general. Let r be a general rational number with $0 < r \leq 1/2$, namely either $r = [m, m_2, \dots, m_k]$, where $m \geq 2$, $m_2 \geq 2$ and $k \geq 3$, or $r = [m, 1, m_3, \dots, m_k]$, where $m \geq 2$ and $k \geq 4$. Suppose on the contrary that there exist two distinct rational numbers $s, s' \in I_1(r) \cup I_2(r)$ such that α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely suppose that Hypothesis A is satisfied. Let \tilde{r} be as in Lemma 3.7. Then by Propositions 9.3 and 9.8, there are two distinct rational numbers $\tilde{s}, \tilde{s}' \in I_1(\tilde{r}) \cup I_2(\tilde{r})$ such that the unoriented loops $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ are homotopic in $S^3 - K(\tilde{r})$. Moreover, there is a reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$, such that none of the degree 4 vertices is mixing. Regardless of the type of r , we put $\tilde{r} = [n_1, \dots, n_t]$, where $t \geq 2$, each $n_i \in \mathbb{Z}_+$ and $n_t \geq 2$. We proceed the proof by induction on $t \geq 2$.

Case 1. $t = 2$, i.e., $\tilde{r} = [n_1, n_2]$. Then \tilde{r} is special, and hence Proposition 10.1 shows that any reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ contains a mixing vertex. This contradicts to the fact observed in the above that there is a reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ which has no mixing vertex.

Case 2. $t = 3$, i.e., $\tilde{r} = [n_1, n_2, n_3]$. If $n_1 \geq 2$ and $n_2 \geq 2$, then \tilde{r} is general and the rational number $\tilde{\tilde{r}} = [n_2 - 1, n_3]$ is as in Case 1. So, this is impossible by the conclusion in Case 1. If $n_1 \geq 2$ and $n_2 = 1$, then \tilde{r} is special and so Proposition 10.1 implies that any reduced conjugacy diagram over $G(K(\tilde{r}))$ for $\alpha_{\tilde{s}}$ and $\alpha_{\tilde{s}'}$ contains a mixing vertex, a contradiction. If $n_1 = 1$, then \tilde{r} is special, because $1 - \tilde{r} = [n_2 + 1, n_3]$ is special; so we obtain a similar contradiction by Proposition 10.1. (To be precise, this rational number does not belong to the list in Proposition 10.1, which is also a contradiction.)

Case 3. $t = 4$, i.e., $\tilde{r} = [n_1, n_2, n_3, n_4]$. If $n_1 \geq 2$, then \tilde{r} is general and the rational number $\tilde{\tilde{r}}$ is as in Case 1 or Case 2 according to whether $n_2 = 1$ or $n_2 \geq 2$. So, this is impossible by the conclusions in Cases 1 and 2. Hence we have $n_1 = 1$ and so let $\tilde{r}' := 1 - \tilde{r} = [n_2 + 1, n_3, n_4]$. If $n_3 \geq 2$, then \tilde{r}' is general and the rational number $\tilde{\tilde{r}}' = [n_3 - 1, n_4]$ is as in Case 1. So, this is impossible by the conclusion in Case 1. If $n_3 = 1$, then \tilde{r} is special, because $1 - \tilde{r} = \tilde{r}'$ is special; so we obtain a contradiction by Proposition 10.1. (In this case, \tilde{r} is as in Proposition 10.1(4).)

Case 4. $t \geq 5$. If $n_1 \geq 2$, then \tilde{r} is general, and the rational number $\tilde{\tilde{r}}$ is as

in the case for $t - 2$ or $t - 1$ according to whether $n_2 = 1$ or $n_2 \geq 2$. Thus this is impossible by the inductive hypothesis. Hence we have $n_1 = 1$ and so let $\tilde{r}' := 1 - \tilde{r} = [n_2 + 1, n_3, \dots, n_t]$. Note that \tilde{r}' is general and the rational number \tilde{r}' is as in the case for $t - 3$ or $t - 2$ according to whether $n_3 = 1$ or $n_3 \geq 2$. Thus this is impossible by the inductive hypothesis.

Main Theorem 2.5 is now completely proved. \square

11. PROOF OF THEOREMS 2.6 AND 2.7

Consider a hyperbolic 2-bridge link $K(r)$ with $0 < r \leq 1/2$, and assume that the loop α_s with $s \in I_1(r) \cup I_2(r)$ is either peripheral or imprimitive. Then, by [8, Lemma 8.2], there is a nontrivial element $w \in G(K(r))$ such that $w \notin \langle u_s \rangle$ and $wu_s w^{-1} = u_s$. This identity cannot hold in $F(a, b)$, since u_s is not a nontrivial cyclic permutation of itself. So by [8, Lemma 8.1], the identity $wu_s w^{-1} = u_s$ in $G(K(r))$ is realized by a nontrivial reduced annular R -diagram, M , with outer and inner labels u_s and u_s^{-1} , respectively. Then M satisfies the assumption of Theorem 3.19 and hence its conclusion.

If $r = [2, n]$ or $[2, 1, n]$ for some $n \geq 2$, then Theorems 2.6 and 2.7 are already proved in [8, Section 8], where all possible diagrams M are described. We note that we can observe that all such diagram contain a mixing vertex of degree 4 (see [8, Figures 15, 16(b), 17(b) and 18]).

If $r = [n, 2]$ or $[n, 1, 2]$ with $n \geq 2$, then the link $K(r)$ is equivalent to the link whose slope is of the previous type, and so Theorems 2.6 and 2.7 in this case are deduced from the results in [8, Section 8].

If $r = [m, n]$ or $[m, 1, n]$ with $m, n \geq 3$, then by the results in Sections 5 and 6, we see that there are no such diagrams. So, any α_s with $s \in I_1(r) \cup I_2(r)$ is neither peripheral nor imprimitive in this case.

Finally, suppose that r is general. Then, by the proof of Propositions 9.3 and 9.8, we can construct from M a non-trivial reduced conjugacy diagram \tilde{M} over $G(K(\tilde{r}))$ with outer and inner labels $u_{\tilde{s}}$ and $u_{\tilde{s}}^{-1}$, respectively, for some $\tilde{s} \in I_1(\tilde{r}) \cup I_2(\tilde{r})$, such that \tilde{M} contains a mixing vertex of degree 4. However, we can inductively show that this is impossible by using the preceding results. Hence, any α_s with $s \in I_1(r) \cup I_2(r)$ is neither peripheral nor imprimitive in this case.

This completes the proof of Theorems 2.6 and 2.7. \square

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